

More than a Lifetime

DISSERTATION

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More than a Lifetime: Towards Precision Predictions for *B*-Meson Decays in the Standard Model and Beyond

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Abstract

This thesis presents a systematic investigation of heavy hadron lifetimes, focusing on the B mesons, as a probe of both Standard Model (SM) dynamics and possible effects of physics beyond the SM (BSM). Within the framework of the Heavy Quark Expansion (HQE), we compute, at leading order in QCD, the complete dimension-seven two-quark contributions, corresponding to the $1/m_b^4$ corrections, for the CKM-favoured non-leptonic decays $b \rightarrow c\bar{u}d$ and $b \rightarrow c\bar{c}s$, providing the full analytical expressions for the corresponding Wilson coefficients. These results, which reveal new matrix elements unique to non-leptonic decays, lay the foundation for future numerical studies and refined SM lifetime predictions. Having extended the SM calculation to the next order in the HQE, we turn to the dimension-six Darwin contribution, whose sizeable numerical impact in the SM motivates its generalization to a model-independent BSM framework. In particular, we derive, at leading order in QCD, the Wilson coefficients up to and including the dimension-six two-quark (Darwin) operators, corresponding to the $1/m_b^3$ corrections in the HQE, for non-leptonic decays induced by a general effective Hamiltonian. We further determine the previously missing dimension-six four-quark weak-annihilation contribution induced by the same general operator basis. Since the SM QCD-penguin operators are contained within this basis, our calculation also yields, as a special case, the previously missing dimension-five and dimension-six contributions induced by the SM QCD-penguin operators, thereby further improving the theoretical precision of HQE predictions. Finally, we explore the phenomenological implications of the B -Mesogenesis model, which links the matter–antimatter asymmetry of the universe and dark matter to exotic B -meson decays. The model is tested through both inclusive and exclusive decay channels: the inclusive analysis, performed within the HQE, quantifies the impact of these new decay modes on the total decay rates and the lifetime ratios $\tau(B^+)/\tau(B_d)$ and $\tau(B_s)/\tau(B_d)$, while the exclusive analysis employs light-cone sum rules to calculate the branching fraction for the exclusive decay $B^+ \rightarrow p\psi$, involving a proton and a dark-sector fermion in the final state. By comparing our predictions with experimental limits from BaBar, Belle, and Belle II, we constrain specific effective operators of the model. Furthermore, by examining power-suppressed terms, we identify regions in the dark-sector mass parameter space where subleading effects exceed the leading dimension-three contributions—offering new insight into the applicability of the operator product expansion within the HQE. Together, these studies advance the theoretical understanding of heavy-hadron lifetimes, strengthen the bridge between precision flavour physics and physics beyond the SM, and open new avenues for probing the interplay between heavy-quark dynamics and the dark sector.

Zusammenfassung

Diese Dissertation präsentiert eine systematische Untersuchung der Lebensdauern schwerer Hadronen mit besonderem Schwerpunkt auf B -Mesonen als Sonden sowohl für die Dynamik des Standardmodells (SM) als auch für mögliche Effekte von Physik jenseits des Standardmodells (BSM). Im Rahmen der Heavy Quark Expansion (HQE) berechnen wir die vollständigen führenden Zwei-Quark-Beiträge der Dimension sieben, entsprechend den $1/m_b^4$ -Korrekturen, für die CKM-begünstigten nichtleptonischen Zerfälle $b \rightarrow c\bar{u}d$ und $b \rightarrow c\bar{c}s$ und bestimmen die vollständigen analytischen Ausdrücke der zugehörigen Wilson-Koeffizienten. Diese Ergebnisse identifizieren neue Matrixelemente, die ausschließlich bei nichtleptonischen Zerfällen auftreten, und bilden die Grundlage für zukünftige numerische Untersuchungen sowie präzisere SM-Vorhersagen der B -Mesonenlebensdauern.

Nach der Erweiterung der SM-Berechnung auf die nächste Ordnung der HQE untersuchen wir den Darwin-Beitrag der Dimension sechs, dessen beträchtlicher numerischer Einfluss im Standardmodell eine Verallgemeinerung auf einen modellunabhängigen BSM-Rahmen motiviert. Insbesondere bestimmen wir auf führender Ordnung die Wilson-Koeffizienten bis einschließlich der Zwei-Quark-Operatoren der Dimension sechs (Darwin-Operatoren), entsprechend den $1/m_b^3$ -Korrekturen der HQE, für nichtleptonische Zerfälle, die durch einen allgemeinen effektiven Hamiltonoperator beschrieben werden. Darüber hinaus berechnen wir erstmals die bislang fehlenden Beiträge der Dimension sechs aus der Vier-Quark-Schwachen-Annihilation für dieselbe allgemeine Operatorbasis. Da die QCD-Penguin-Operatoren des Standardmodells in dieser Basis enthalten sind, umfasst unsere Berechnung als Spezialfall zugleich die bislang fehlenden Beiträge der Dimension fünf und sechs dieser Operatoren und verbessert damit die theoretische Präzision der HQE-Vorhersagen.

Abschließend untersuchen wir die phänomenologischen Konsequenzen des Modells B -Mesogenesis, das die Materie-Antimaterie-Asymmetrie des Universums und Dunkle Materie mit exotischen B -Mesonen-Zerfällen verknüpft. Das Modell wird anhand inklusiver und exklusiver Zerfallskanäle untersucht. Die inklusive HQE-Analyse quantifiziert den Einfluss dieser Zerfälle auf die Gesamtzerfallsraten sowie die Lebensdauerverhältnisse $\tau(B^+)/\tau(B_d)$ und $\tau(B_s)/\tau(B_d)$. Die exklusive Analyse basiert auf Light-Cone Sum Rules (LCSR) und berechnet Verzweungsverhältnisse in baryonische Endzustände mit Fermionen des dunklen Sektors. Durch den Vergleich unserer Vorhersagen mit den experimentellen Grenzen von BaBar, Belle und Belle II schränken wir spezifische effektive Operatoren des Modells ein. Darüber hinaus identifizieren wir durch die Untersuchung potenzunterdrückter Beiträge Bereiche des Massenparameters des dunklen Sektors, in denen subführende Beiträge die führenden Terme der Dimension drei übertreffen. Dies liefert neue Erkennt-

nisse über den Anwendungsbereich der Operatorproduktentwicklung innerhalb der HQE. Insgesamt vertiefen diese Arbeiten das Verständnis der Lebensdauern schwerer Hadronen, stärken die Verbindung zwischen Präzisions-Flavourphysik und Physik jenseits des Standardmodells und eröffnen neue Perspektiven für die Untersuchung des Zusammenspiels von Schwerquarkdynamik und dem dunklen Sektor.

Publications

This thesis is based on the following publications:

1. A. Lenz, A. Mohamed, and Z. Wüthrich, *Constraining B -mesogenesis models with inclusive and exclusive decays*, JHEP **08** (2025) 141, [arXiv:2412.14947](#).
2. A. Mohamed, *Are Subleading Effects Really Subleading? B -Meson Decays in Mesogenesis*, JHEP **06** (2026) 219, [arXiv:2511.04858](#).
3. M. Lang, A. Lenz, A. Mohamed, M. L. Piscopo, and A. V. Rusov, *B -meson decay width up to $1/m_b^3$ corrections within and beyond the Standard Model*, JHEP **05** (2026) 100, [arXiv:2512.14635](#).
4. A. Biswas, A. Khodjamirian, and A. Mohamed, *Light-cone sum rules with B -meson distribution amplitudes for the $B \rightarrow p$ form factors in B -mesogenesis models*, Phys. Rev. D **113** (2026), no. 11, 116035, [arXiv:2603.01723](#).

In addition, this thesis includes material from the following manuscript currently in preparation:

- A. Mohamed, M. Lang and A. Lenz, *HQE at $\mathcal{O}(1/m_b^4)$: Two-Quark Operators Contribution to Non-Leptonic B Decays* (in preparation).

The correspondence between the chapters of the thesis and the above publications is as follows:

- Chapter 2 is based on the manuscript currently in preparation.
- Chapter 3 is based primarily on Publication 3.
- Chapter 4 is based primarily on Publications 1, 2, and 4.

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For those who search without guarantee of discovery, and who persevere for the simple love of the work. This thesis is dedicated to that spirit, and to the people who nurtured it in me.

“It is thus the duty of the man who studies the writings of scientists, if learning the truth is his goal, to make himself an enemy of all that he reads, and, applying his mind to the core and margins of its content, attack it from every side.”

— Ibn al-Haytham

Introduction

The Standard Model (SM) of particle physics [1–3] is a remarkably successful theory that describes the electromagnetic, weak, and strong interactions of elementary particles. It has withstood decades of experimental scrutiny, culminating in the discovery of the Higgs boson by the ATLAS [4] and CMS [5] collaborations in 2012, which confirmed the last missing element of the SM and provided direct evidence for the origin of particle masses via the Brout-Englert-Higgs mechanism [6–8].

Despite its success, the SM is widely regarded as incomplete. It does not incorporate gravity and fails to explain several key phenomena, such as the baryon asymmetry of the universe [9] and the nature of dark matter [10, 11] and dark energy [12, 13]. Moreover, as experimental precision continues to improve, rare B -meson decays provide some of the most sensitive tests of the SM. Examples include measurements of angular observables in $b \rightarrow s\ell^+\ell^-$ transitions and tests of lepton flavour universality in rare semileptonic B decays, which have motivated extensive theoretical and experimental scrutiny in recent years, see e.g. [14, 15].

To explore these unresolved mysteries, two complementary approaches can be pursued. One is to refine SM predictions and compare them with increasingly precise experimental measurements. The other is to construct and investigate extensions of the SM that could provide solutions to these open questions and derive their phenomenological consequences, see e.g. [16]. The former approach is valuable in its own right, as it enables increasingly stringent tests of the SM and may reveal deviations that point towards new physics. Meanwhile, the latter provides concrete theoretical frameworks whose predictions can be tested both indirectly through precision observables and directly through the production of new particles at current and future collider experiments. Furthermore, precise SM predictions play a central role in interpreting indirect searches and constraining beyond the SM (BSM) scenarios by allowing potential new-physics effects to be disentangled from the SM contribution. One of the most powerful arenas for pursuing both approaches is the field of flavour physics.

Flavour physics continues to be a highly active and central field in modern particle physics. It provides a unique window into testing the SM at high precision and probing potential new physics beyond it, see e.g. the review [17]. The field is driven by a broad experimental programme, with dedicated measurements performed at LHCb, Belle II, and BESIII, complemented by the ATLAS and CMS experiments at the LHC. In particular, the study of B mesons plays a crucial role due to the rich flavour structure of the quark sector and the wealth of precise experimental data available.

While the SM successfully describes flavour-changing processes through the Cabibbo–

Kobayashi–Maskawa (CKM) mechanism [18, 19], it does not explain the origin of the fermion mass hierarchies, the pattern of quark mixing [20], or the observed matter–antimatter asymmetry in the Universe. Furthermore, flavour-changing neutral currents are highly suppressed within the SM through the Glashow–Iliopoulos–Maiani (GIM) mechanism [21], making flavour observables particularly sensitive probes of new physics. Indeed, many BSM scenarios introduce additional sources of flavour violation, which can manifest themselves as subtle deviations from SM predictions, as discussed in, for example, Ref. [22]. Precision measurements of observables such as branching ratios and B -meson lifetimes therefore provide complementary indirect probes of physics beyond the SM and are sensitive to new particles that may lie beyond the direct reach of current collider experiments. Their interpretation relies on equally precise theoretical predictions, motivating continued efforts to refine SM calculations and to explore possible signatures of new physics in the flavour sector. In this thesis, we pursue two complementary approaches. On the SM side, we improve the theoretical description of inclusive non-leptonic B -meson decays within the framework of the Heavy Quark Expansion (HQE) [23–26]. Specifically, we determine, at leading-order QCD, the matching coefficients of the dimension-seven two-quark operators, corresponding to the $1/m_b^4$ corrections, for the CKM-favoured transitions $b \rightarrow \bar{c}ud$ and $b \rightarrow \bar{c}\bar{s}s$. Furthermore, we calculate the previously missing leading-order matching coefficients of the dimension-five and dimension-six two-quark operators generated by the QCD-penguin operators [27–29], thereby further improving the theoretical precision and completeness of HQE predictions. On the BSM side, we extend the HQE for inclusive non-leptonic B decays to a generalised effective Hamiltonian by deriving the leading-order Wilson coefficients of all two-quark contributions up to order $1/m_b^3$. In addition, we determine the previously missing dimension-six four-quark weak-annihilation contribution for the same generalised effective Hamiltonian. Complementing the BSM model-independent analysis, we also perform a model-specific study of the B -Mesogenesis scenario [30–33], which aims to explain the matter–antimatter asymmetry of the Universe and the nature of dark matter. This model predicts exotic B -meson decays into a dark-sector fermion ψ , which would manifest as missing energy in detectors. We study the impact of these decays on phenomenological observables of the B meson in both inclusive and exclusive channels. The inclusive analysis is carried out using the HQE, while exclusive decays are investigated within the framework of Light-Cone Sum Rules (LCSR), discussed in Section 1.4.

The calculations presented in this thesis rely on two robust theoretical frameworks. From a theoretical standpoint, the HQE has established itself as a powerful framework for studying inclusive decays and lifetimes of heavy hadrons. Over the past decades, the expansion in inverse powers of the b -quark mass has undergone significant development, leading to increasingly precise predictions at higher orders. As of the writing of this thesis, the state of the art includes the next-to-leading order (NLO) calculations of the dimension-five contributions Γ_5 in non-leptonic decays, both for the $b \rightarrow \bar{c}s$ [34] and $b \rightarrow \bar{c}ud$ [35] channels, the next-to-next-to-leading order (NNLO) determination of the leading dimension-three contribution Γ_3 [36], and the NNLO calculation of the dimension-six four-quark spectator contribution $\tilde{\Gamma}_6$ [37]. Significant progress has also been made on the non-perturbative side. In particu-

lar, Heavy Quark Effective Theory (HQET) sum-rule determinations of the matrix elements of dimension-six $\Delta B = 0$ four-quark operators, both within the SM and for a general BSM operator basis, have recently become available [38]. Furthermore, the first lattice-QCD determinations of dimension-six four-quark operator matrix elements using the gradient-flow method have been achieved [39, 40]. Although currently restricted to charmed mesons, they represent an important step towards first-principles determinations of HQE matrix elements.

The precision of HQE predictions has improved substantially in recent years and, for several observables, is now comparable to the experimental precision. For many of these observables, the remaining theoretical uncertainties are dominated by the non-perturbative matrix elements entering the HQE, highlighting the importance of constraining them from experimental data and determining them using first-principles methods such as lattice QCD or QCD sum rules. A detailed discussion of the HQE formalism and its applications within the SM can be found in Refs. [41, 42].

In the context of exclusive decays, the LCSR approach has proven to be a reliable and versatile tool, see e.g. the review [43]. It combines perturbative QCD with non-perturbative hadronic input and allows for systematic improvements in accuracy through the inclusion of higher-twist corrections and radiative effects [44, 45].

The thesis is divided as follows. [Chapter 1](#) introduces the theoretical framework used throughout this work. It begins with the formulation of the Effective Weak Hamiltonian, which provides the foundation for describing weak decays of heavy hadrons. The HQE is then presented as a systematic tool to compute inclusive observables in powers of $1/m_b$. The chapter further outlines the Background Field Method and the use of the Fock-Schwinger gauge, which are essential for generating the operator product expansion at the field-theoretical level. Finally, the framework of LCSR is introduced, providing the basis for the calculation of exclusive decay processes in later chapters. In [Chapter 2](#), we investigate the dimension-seven sector of the HQE for inclusive non-leptonic B -meson decays. We derive the complete leading order dimension-seven two-quark contributions, corresponding to the $1/m_b^4$ corrections, for the CKM-favoured decay channels $b \rightarrow c\bar{u}d$ and $b \rightarrow c\bar{c}s$, providing the full analytical expressions for the corresponding Wilson coefficients. In [Chapter 3](#), we develop a model-independent extension of the HQE for inclusive non-leptonic B decays based on a generalised effective Hamiltonian. Within this framework, we derive the leading-order Wilson coefficients of all two-quark contributions up to order $1/m_b^3$ and determine the previously missing dimension-six four-quark weak-annihilation contribution. We further show that the SM QCD-penguin operators arise as a special case of the general operator basis, allowing us to determine their previously missing dimension-five and dimension-six contributions. In [Chapter 4](#), we present a comprehensive phenomenological study of the B -Mesogenesis scenario. On the inclusive side, we employ the HQE to determine the impact of baryon-number-violating operators on B -meson lifetimes and derive constraints from lifetime ratios and collider searches. On the exclusive side, we compute the hadronic matrix elements for $B^+ \rightarrow p\psi$ decays using LCSR. Combining both approaches, we obtain stringent constraints on the viable parameter space of the model. Finally, [Chapter 5](#) summarizes the main results and discusses directions for future work.

Chapter 1

Theoretical Framework

“Our job requires a great deal of resilience, and losing is part of the game.”

— Alexander Lenz

1.1 Effective Weak Hamiltonian

One of the characteristic features of weak interactions is the presence of widely separated energy scales. While weak decays occur at energies of order a few GeV, they are mediated by much heavier particles, such as the electroweak gauge bosons with masses of about 80–90 GeV and the top quark with a mass of approximately 173 GeV. Consequently, many weak-decay processes occur at energy scales far below the masses of the particles that mediate the underlying interactions. A complete description of such processes would require treating simultaneously both the light degrees of freedom that actively participate in the dynamics and much heavier particles. Moreover, the presence of these widely separated scales gives rise to large logarithms of the form $\ln(M_W/\mu)$ in perturbative calculations, where M_W denotes the W -boson mass and μ the characteristic low-energy renormalisation scale of the process. Such logarithms can spoil the convergence of fixed-order perturbation theory.

Effective Field Theories (EFTs) [46, 47], see also [48, 49], provide a systematic framework to deal with this situation. By “integrating out” the heavy particles—i.e., removing them from the spectrum of active degrees of freedom—we obtain an effective theory that describes the low-energy physics in terms of local operators composed of light fields. The effects of the heavy particles are not lost but are encoded in the so-called Wilson coefficients, which multiply these operators and capture the short-distance (high-energy) physics. Importantly, the effective theory provides a systematic expansion in inverse powers of the heavy scale, allowing higher-order corrections to be incorporated in a controlled, order-by-order manner. Moreover, the separation of scales permits the resummation of the large logarithms through the renormalisation-group evolution of the Wilson coefficients, thereby restoring the convergence of the perturbative expansion.

For weak decays of hadrons, the heavy particles that are integrated out are primarily the electroweak gauge bosons and, depending on the process, the top quark. As a result, the underlying electroweak interaction is replaced by an effective weak Hamiltonian consisting of local operators multiplied by Wilson coefficients.

EFTs thus allow us to simplify complex interactions, perform consistent expansions in small parameters (such as E/M_W), where E denotes the characteristic energy or momentum transfer of the process, and compute observables with controlled theoretical uncertainties. At the same time, they offer a model-independent

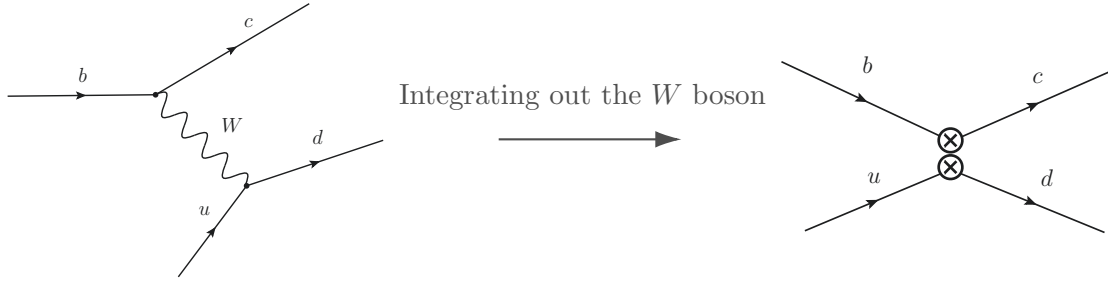


Figure 1.1: Illustration of the construction of the effective weak Hamiltonian for the transition $b \rightarrow c\bar{u}d$. At energies well below the electroweak scale, the W boson is integrated out, replacing the full SM interaction by a local four-fermion interaction.

description of possible BSM effects by parametrizing short-distance physics in terms of higher-dimensional operators and their corresponding Wilson coefficients, without requiring knowledge of the underlying ultraviolet (UV) completion. This framework is the foundational first step for all subsequent calculations of weak decay amplitudes in this thesis.

It will be instructive to illustrate the transition from the full Lagrangian to the effective Hamiltonian with a concrete example. Consider the tree-level weak decay $b \rightarrow c\bar{u}d$, which proceeds via the exchange of a virtual W boson in the SM, see Figure 1.1. The second-order contribution to the $\hat{S}$ -operator from the weak interaction is (see, Appendix A)

$$\hat{S}^{(2)} = -\frac{g^2}{2} V_{cb} V_{ud}^* \int d^4x d^4y T \left\{ [\bar{c}^i(x) \gamma^\mu P_L b^i(x)] W_\mu^-(x) [\bar{d}^j(y) \gamma^\nu P_L u^j(y)] W_\nu^+(y) \right\}, \quad (1.1)$$

where i and j are colour indices. After performing the time-ordered product to contract the charged gauge fields, one obtains the W -boson propagator in unitary gauge. Explicitly in momentum space ¹,

$$\langle 0 | T \{ W_\mu^-(x) W_\nu^+(y) \} | 0 \rangle = \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 - M_W^2 + i\epsilon} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2} \right) e^{-iq \cdot (x-y)}. \quad (1.2)$$

Performing the x - and y -integrations yields the overall momentum-conserving delta function and the momentum-space amplitude

$$iT_{fi} = \frac{g^2}{2} V_{cb} V_{ud}^* \frac{1}{q^2 - M_W^2} [\bar{c}^i \gamma^\mu P_L b^i] [\bar{d}^j \gamma_\mu P_L u^j] \quad (1.3)$$

At low energies, where $q^2 \ll M_W^2$, the propagator can be expanded as:

$$\frac{1}{q^2 - M_W^2} \approx -\frac{1}{M_W^2} \left(1 + \frac{q^2}{M_W^2} + \dots \right), \quad (1.4)$$

keeping only the leading term, the amplitude reduces to

$$iT_{fi} = -i \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* [\bar{c}^i \gamma^\mu P_L b^i] [\bar{d}^j \gamma_\mu P_L u^j] + \mathcal{O}(q^2/M_W^2) \quad (1.5)$$

¹In the low-energy limit, the longitudinal term proportional to $q_\mu q_\nu / M_W^2$ gives a suppressed contribution and is therefore neglected.

where $G_F = \frac{\sqrt{2}g^2}{8M_W^2}$ is the Fermi constant. To construct the corresponding low-energy description, one introduces an effective interaction Hamiltonian $\mathcal{H}_{\text{eff}}$ such that the $\hat{S}$ -operator²,

$$\hat{S} = T \exp \left[-i \int d^4x \mathcal{H}_{\text{eff}}(x) \right], \quad (1.6)$$

reproduces the same transition amplitude in Eq. (1.5) as obtained from the full theory. In this framework, the effect of the heavy W -boson exchange is encoded in a local four-fermion operator, and the amplitude arises already at *first order* in $\mathcal{H}_{\text{eff}}$, rather than from the second-order weak interaction in the full Lagrangian. This transition from the full theory to the effective description is illustrated schematically in Figure 1.1. From Eq. (1.5), the effective Hamiltonian then reads

$$\mathcal{H}_{\text{eff}}^{b \rightarrow c\bar{u}d}(x) = \frac{4G_F}{\sqrt{2}} V_{cb}V_{ud}^* [\bar{c}^i(x)\gamma^\mu P_L b^i(x)] [\bar{d}^j(x)\gamma_\mu P_L u^j(x)]. \quad (1.7)$$

This local four-fermion interaction encapsulates the short-distance effects of the W -boson exchange and constitutes the leading-order term in the weak effective theory.

The operator in Eq. (1.7) represents the first element in an infinite tower of higher-dimensional operators, each suppressed by powers of $1/M_W$ and multiplied by short-distance Wilson coefficients that encode the effects of heavy degrees of freedom. The general form of the Effective Weak Hamiltonian for a given decay process can be written as

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_i V^{\text{CKM}} C_i(\mu) O_i(\mu), \quad (1.8)$$

where V^{CKM} denotes the appropriate combination of CKM matrix elements associated with the process, $C_i(\mu)$ are the *Wilson coefficients*, encoding the short-distance dynamics above the renormalisation scale μ ; they are process-independent and can be computed perturbatively, $O_i(\mu)$ are the *local operators*, built from light (quark, gluon, photon) fields, which capture the long-distance physics. Their hadronic matrix elements must be evaluated using non-perturbative methods.

The separation of short- and long-distance dynamics in the effective Hamiltonian introduces an explicit dependence on the renormalisation scale μ . To ensure that physical observables remain independent of this arbitrary scale, the Wilson coefficients and operator matrix elements must evolve consistently under the Renormalisation Group.

As illustrated above for the $b \rightarrow c\bar{u}d$ transition, the construction of the effective theory begins with a *matching* calculation, in which the Green's functions (or scattering amplitudes) computed in the full SM are required to agree with those of the effective theory at a matching scale $\mu_W \sim M_W$. In this way, the effects of the heavy degrees of freedom, such as the W boson and the top quark, are absorbed into the Wilson coefficients, while the heavy particles themselves are removed from the low-energy theory. The resulting Wilson coefficients $C_i(\mu_W)$ are then evolved down to a characteristic hadronic scale, $\mu_{\text{had}} \sim m_b$ by solving the Renormalisation Group Equations (RGEs). It is precisely this renormalisation-group evolution that resums

²The minus sign in the exponent arises from the simple relation $\mathcal{H}_{\text{eff}} = -\mathcal{L}_{\text{eff}}$.

the large logarithms of the form $\ln(M_W^2/m_b^2)$ arising from the hierarchy of scales discussed above, thereby restoring the convergence of the perturbative expansion and ensuring a consistent connection between the short-distance and low-energy descriptions. For detailed discussions of the renormalisation-group evolution and operator mixing in weak effective theories, see Refs. [29, 50].

Although the matching between the full SM and the effective theory can be systematically improved by including higher-order perturbative QCD corrections, the tree-level matching presented above is sufficient to illustrate the underlying procedure. Since the calculations in this thesis are consistently performed at leading order in QCD, we do not discuss the higher-order matching in detail. Nevertheless, the numerical analyses throughout this thesis employ Wilson coefficients evaluated at the appropriate perturbative precision available in the literature.

Having introduced the general structure of the effective weak Hamiltonian, we now present its explicit form in the SM. For non-leptonic b -quark decays, it is given by [29]

$$\mathcal{H}_{\text{eff}}^{\text{SM}} = \frac{4G_F}{\sqrt{2}} \sum_{q_3=d,s} \left\{ \sum_{q_1,q_2=u,c} V_{q_1 b} V_{q_2 q_3}^* \left[C_1(\mu_b) Q_1^{q_1 q_2 q_3} + C_2(\mu_b) Q_2^{q_1 q_2 q_3} \right] - V_{tb} V_{tq_3}^* \sum_{k=3}^6 C_k(\mu_b) \sum_{q=u,d,s,c} Q_k^{qq_3} \right\} + \text{h.c.}, \quad (1.9)$$

where G_F is the Fermi constant and $V_{qq'}$ denote the elements of the CKM matrix. The operator basis relevant for the non-leptonic $|\Delta B| = 1$ transitions in Eq. (1.9) comprises two current-current operators, $Q_{1,2}^{q_1 q_2 q_3}$, together with the QCD-penguin operators $Q_{3,\dots,6}^{qq_3}$. We adopt the following conventions:

$$Q_1^{q_1 q_2 q_3} = (\bar{q}_1^i \gamma_\mu P_L b^i)(\bar{q}_3^j \gamma^\mu P_L q_2^j), \quad Q_2^{q_1 q_2 q_3} = (\bar{q}_1^i \gamma_\mu P_L b^j)(\bar{q}_3^j \gamma^\mu P_L q_2^i), \quad (1.10)$$

for the current-current sector, while the QCD-penguin operators are

$$Q_3^{qq_3} = (\bar{q}_3^i \gamma_\mu P_L b^i)(\bar{q}^j \gamma^\mu P_L q^j), \quad Q_4^{qq_3} = (\bar{q}_3^i \gamma_\mu P_L b^j)(\bar{q}^j \gamma^\mu P_L q^i), \quad (1.11)$$

$$Q_5^{qq_3} = (\bar{q}_3^i \gamma_\mu P_L b^i)(\bar{q}^j \gamma^\mu P_R q^j), \quad Q_6^{qq_3} = (\bar{q}_3^i \gamma_\mu P_L b^j)(\bar{q}^j \gamma^\mu P_R q^i). \quad (1.12)$$

The indices $i, j = 1, 2, 3$ denote fundamental $SU(3)_c$ colour indices, and the left- and right-handed chiral projectors are defined by $P_{L,R} = (1 \mp \gamma_5)/2$.

In Eq. (1.9), the Wilson coefficients $C_k(\mu_b)$, with $k = 1, \dots, 6$, are evaluated at the renormalisation scale $\mu_b \sim m_b$ and are known up to NNLO in QCD [51]. In the SM, these coefficients are real-valued, whereas in generic extensions with heavy new degrees of freedom integrated out at a scale Λ_{NP} , they can in general be complex-valued. Additional operators, such as the electroweak penguin operators Q_7 – Q_{10} and the electromagnetic and chromomagnetic dipole operators $Q_{7\gamma}$ and Q_{8g} , are omitted for brevity, as they only give subleading contributions to the processes studied in this work.

While the discussion above introduces the SM $|\Delta B| = 1$ weak effective Hamiltonian, this thesis will employ different effective Hamiltonians depending on the

physical application. In the study of model-independent BSM contributions in [Chapter 3](#), extended operator bases with different Dirac structures will be introduced. In [Chapter 4](#), the analysis of B -Mesogenesis employs the effective Hamiltonian with charge-conjugated fields. Each Hamiltonian is presented explicitly in the corresponding chapter.

1.2 Heavy Quark Expansion

All inclusive B -meson decay widths considered in this work are calculated within the framework of the HQE. This section introduces the HQE formalism, its application to B -meson decays, and the challenges associated with the determination of non-perturbative matrix elements.

The HQE is a systematic effective field theory approach for describing inclusive decay rates and lifetimes of hadrons containing a heavy quark, such as B mesons. It is based on an operator product expansion (OPE) [[47](#), [52](#), [53](#)] in inverse powers of the heavy-quark mass, $1/m_b$, which separates short-distance effects—encoded in perturbatively calculable Wilson coefficients—from long-distance contributions, contained in non-perturbative hadronic matrix elements.

Building upon the Effective Weak Hamiltonian introduced in the previous section, the HQE provides a framework in which the total decay rate of a heavy hadron can be expressed as a series of local operators of increasing dimension, suppressed by corresponding powers of $1/m_b$. This formalism allows for precise theoretical predictions of observables such as total B -meson decay widths and lifetime ratios, e.g., $\tau(B^+)/\tau(B_d)$ and $\tau(B_s)/\tau(B_d)$, by systematically incorporating both perturbative and non-perturbative QCD effects.

The HQE exploits the fact that the b -quark mass is much larger than the typical QCD scale ($m_b \gg \Lambda_{\text{QCD}}$), allowing the inclusive decay of a B meson to be approximated as a free b -quark decay, corrected by systematic non-perturbative contributions from the light degrees of freedom (quarks and gluons) inside the hadron.

According to the optical theorem, the inclusive decay rate can be expressed in terms of the imaginary part of the forward scattering amplitude:

$$\Gamma(B \rightarrow X) = \frac{1}{2m_B} \text{Im} \langle B | i \int d^4x \text{T}\{\mathcal{H}_{\text{eff}}(x), \mathcal{H}_{\text{eff}}(0)\} | B \rangle, \quad (1.13)$$

where X denotes the sum over all hadronic final states accessible via the effective Hamiltonian, and T denotes time ordering. Expanding the non-local time-ordered product into a series of local operators through the OPE yields a systematic expansion of the total decay rate in inverse powers of the heavy-quark mass:

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \mathcal{O}_7 \rangle}{m_b^4} + \dots + 16\pi^2 \left[\tilde{\Gamma}_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \dots \right]. \quad (1.14)$$

Here, the terms proportional to Γ_3 , Γ_5 , Γ_6 , $\tilde{\Gamma}_6$, and so on, correspond to contributions from local $\Delta B = 0$ operators of increasing mass dimension, namely dimension-three, dimension-five, dimension-six two-quark, dimension-six four-quark

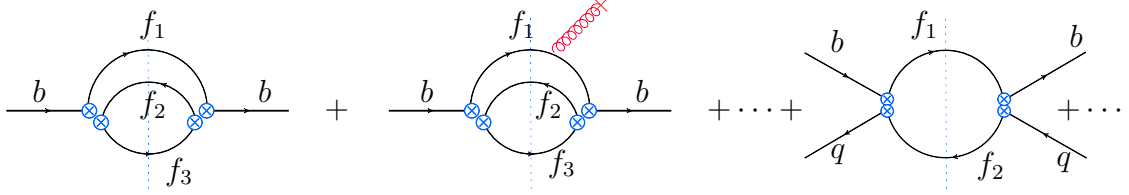


Figure 1.2: Diagrammatic representation of the HQE for B -meson decays. The two-quark and four-quark contributions originate from two-loop and one-loop forward-scattering diagrams, respectively. Crossed vertices correspond to insertions of operators from the effective $|\Delta B| = 1$ Hamiltonian. The vertical dashed line denotes the cut used to obtain the imaginary part of the forward amplitude. The particles f_1 , f_2 , and f_3 represent generic fermionic decay products of the b quark.

operators, and higher-dimensional operators, whose expectation values encode non-perturbative information about the internal structure of the B meson.

The non-tilde operators, $\mathcal{O}_i$, are two-quark operators that describe corrections to the free heavy-quark decay, such as kinetic and chromomagnetic effects. In contrast, the tilde operators, $\tilde{\mathcal{O}}_i$, are four-quark operators that account for spectator interactions between the heavy b -quark and the light valence quarks within the meson. A diagrammatic illustration of the HQE is shown in Figure 1.2. The spectator contributions, represented by $\tilde{\Gamma}_i$, are enhanced by a factor of $16\pi^2$, reflecting the transition from the two-loop topology of the non-spectator diagrams to a one-loop topology in the corresponding forward-scattering diagrams. This phase-space enhancement implies that, despite entering at order $1/m_b^3$, spectator effects can be numerically comparable to or even dominate over some lower-order non-spectator terms. Consequently, they play a crucial role in explaining the observed differences in lifetimes between B^+ , B_d , and B_s mesons.

At leading order, the HQE starts with the dimension-3 operator $\mathcal{O}_3 = \bar{b}b$, which corresponds to the free b -quark decay. Its matrix element is normalized to unity up to small corrections, so it does not induce any lifetime differences among B mesons. The dimension-4 operator, $\mathcal{O}_4 = \bar{b}i\not{D}b$, does not contribute independently because it can be reduced to the leading operator by the equations of motion of the heavy quark, and is therefore absent in the final expansion. At order $1/m_b^2$, two dimension-5 operators appear: the kinetic operator $\mathcal{O}_\pi = \bar{b}(iD)^2b$ and the chromomagnetic operator $\mathcal{O}_G = \frac{g_s}{2}\bar{b}\sigma_{\mu\nu}G^{\mu\nu}b$. Their matrix elements parameterize the motion of the heavy quark inside the hadron and its interaction with the gluon field, respectively. At order $1/m_b^3$, one encounters the dimension-6 contributions. These include both two-quark operators, such as the Darwin and spin-orbit terms, and the four-quark operators that describe spectator effects. The Darwin operator encodes information about the spatial variation of the gluon field inside the hadron, while the spin-orbit term represents the interaction between the heavy-quark spin and the background chromomagnetic field. The four-quark operators, on the other hand, describe nonperturbative corrections sensitive to the flavour and colour of the spectator quark. These

contributions dominate the lifetime differences between B^+ , B_d , and B_s mesons.

Going beyond this order, the HQE includes dimension-7 operators, which arise at $\mathcal{O}(1/m_b^4)$. While numerically suppressed by an additional power of $1/m_b$, such contributions can become relevant in precision studies of inclusive observables and play a central role in the analyses presented later in this thesis.

The HQE framework is general and not limited to SM calculations. In [Chapter 3](#), it is applied to include BSM contributions, such as those affecting the Darwin term, in a model-independent way. [Chapter 4](#) applies the HQE to the study of B -Mesogenesis, where decays into a dark-sector fermion ψ contribute to the total inclusive decay rate and modify the B -meson lifetime ratios. Therefore, in this work, we apply the HQE in both SM and BSM contexts, where all perturbative contributions, Γ_i and $\tilde{\Gamma}_i$, associated with the operators shown in [Eq. \(1.14\)](#), are explicitly calculated. In both cases, however, the operator expansion remains the same in structure, with additional operators³ or modified Wilson coefficients capturing the effects of new physics. All operators that contribute to the observables studied here are presented explicitly along with their Wilson coefficients in the sections where they are relevant.

1.3 Quark Propagator in an External Gluon Field

The evaluation of many quantities in the HQE and LCSR requires the quark propagator in the presence of an external gluon field. A systematic expansion of this propagator can be obtained by treating the gluon field as a slowly varying classical background. This approach was pioneered in the context of QCD sum rules by Shifman, Vainshtein, and Zakharov [\[54\]](#), where it provided the foundation for the OPE in external gluonic fields.

A particularly convenient choice is the Fock–Schwinger (FS) gauge [\[55, 56\]](#), in which the gauge potential is expressed directly in terms of the gluon field-strength tensor and its covariant derivatives. This establishes a transparent connection between quark-level Feynman diagrams and the local operators appearing in the HQE. The same formalism also forms the basis of the derivation of the LCSR discussed in the following section.

The expansion of the quark propagator relies on the hierarchy of scales between the soft external gluon field and the hard momentum carried by the propagating quark. The characteristic momentum carried by the external gluon field is of order Λ_{QCD} , which is small compared to the virtuality or external momentum of the quark line appearing in the short-distance expansion. Consequently, the gluon field can be treated as a slowly varying, or “soft” background, while the quark propagator is expanded perturbatively in powers of this background field. To implement the expansion, we begin by recalling the definition of the free quark propagator,

$$\langle 0 | T \{ \psi(x), \bar{\psi}(y) \} | 0 \rangle = i S_0(x, y), \quad (1.15)$$

³Here, additional operators refer to operators in the effective weak Hamiltonian. New-physics contributions are matched onto the same HQE operator basis defined in [Section B.1](#).

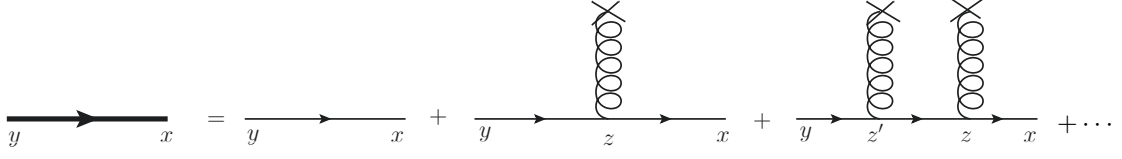


Figure 1.3: Diagrammatic representation of a quark propagating from y to x in an external gluon field, with zero, one, two, $\dots$ soft-gluon insertions.

with

$$S_0(x, y) = \int \frac{d^4 p}{(2\pi)^4} \mathcal{S}_0(p) e^{-ip \cdot (x-y)}, \quad \mathcal{S}_0(p) = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}, \quad (1.16)$$

which describes the propagation of a quark in the absence of any gluonic background. When a soft background gluon field is present, the full propagator can be expressed as an expansion around the free propagator,

$$iS(x, y) = iS_0(x, y) + iS_1(x, y) + iS_2(x, y) + \dots \quad (1.17)$$

$S_1(x, y)$ and $S_2(x, y)$ denote the first and second order corrections, describing the interaction of the quark with one and two gluon fields, respectively, while the ellipsis stand for higher order terms with more than two gluons. Explicitly,

$$iS_1(x, y) = \int d^4 z \, iS_0(x - z) \, i\mathcal{A}(z) \, iS_0(z - y). \quad (1.18)$$

$$iS_2(x, y) = \int d^4 z d^4 z' \, iS_0(x - z) \, i\mathcal{A}(z) \, iS_0(z - z') \, i\mathcal{A}(z') \, iS_0(z' - y). \quad (1.19)$$

A schematic representation of [Eq. \(1.17\)](#) is shown in [Figure 1.3](#).

In the FS gauge, defined by the condition $x^\mu A_\mu(x) = 0$, the gauge potential at a point x can be expressed in terms of the field-strength tensor and its covariant derivatives at the origin as

$$A_\mu^a(x) = \frac{1}{2} x^\rho G_{\rho\mu}^a(0) + \frac{1}{3} x^\rho x^\sigma D_\sigma G_{\rho\mu}^a(0) + \frac{1}{8} x^\rho x^\sigma x^\eta D_\sigma D_\eta G_{\rho\mu}^a(0) + \dots, \quad (1.20)$$

where A_μ^a denotes the gluon field, $G_{\mu\nu}^a$ the gluon field-strength tensor, and D_μ the covariant derivative. Their explicit definitions are collected in [Appendix A](#). The ellipsis denotes terms involving higher covariant derivatives. This representation ensures manifest gauge covariance of the expansion and makes explicit the connection between the propagator corrections and the local gluonic operators appearing in the OPE. A notable feature of the FS gauge is that translational invariance is no longer manifest—the expansion is fixed around the reference point (commonly chosen as $x_0 = 0$) where the field-strength tensor is evaluated.

A variant of the background field method in FS gauge was first introduced in Refs. [\[57–59\]](#). For comprehensive discussions of the technique and its applications, we refer the reader to the lecture notes [\[60\]](#), the technical review [\[61\]](#), and the references therein. For the purposes of this work, we present the explicit form of the quark propagator to the accuracy required, while a full derivation, following the same procedure without fundamentally new steps, is available in Ref. [\[62\]](#).

In this work, the quark propagator will be expanded to generate operators up to dimension seven. This requires including terms with up to two covariant derivatives acting on the gluon field-strength tensor, as well as contributions containing two field-strength tensors.

At this stage it is useful to note that, owing to the purely left-handed ($V - A$) structure of the SM current-current operators, only propagator terms containing an odd number of Dirac matrices contribute to the matching. Contributions involving an even number of gamma matrices vanish after being contracted with the weak vertices and therefore do not contribute to the two-quark matching coefficients. Consequently, throughout this chapter only the terms with an odd number of Dirac matrices are retained in the propagator expansion.

The quark propagator can be written in momentum space as

$$S(x, 0) = \int \frac{d^4 p}{(2\pi)^4} \mathcal{S}(p) e^{-ip \cdot x}, \quad (1.21)$$

with

$$\mathcal{S}(p) = \mathcal{S}_0(p) + \mathcal{S}_1(p) + \mathcal{S}_2(p) + \dots, \quad (1.22)$$

where $\mathcal{S}_0(p)$ denotes the free propagator, while $\mathcal{S}_1(p)$ and $\mathcal{S}_2(p)$ contain one and two insertions of the background gluon field, respectively. The expression for $\mathcal{S}_1(p)$ up to terms involving one covariant derivative on the field-strength tensor is known e.g. from Refs. [62, 63]. Here, it is extended to include the contributions proportional to $D_\sigma D_\eta G_{\rho\mu}$ required for the dimension-seven matching, yielding

$$\begin{aligned} \mathcal{S}_1(p) = & \frac{\tilde{G}_{\sigma\eta} p^\sigma \gamma^\eta \gamma^5}{(p^2 - m^2 + i\varepsilon)^2} - \frac{2}{3} \frac{D_\alpha G_{\rho\mu} \gamma^\mu p^\alpha p^\rho}{(p^2 - m^2 + i\varepsilon)^3} + 2i \frac{D_\alpha \tilde{G}_{\tau\eta} p^\alpha p^\tau \gamma^\eta \gamma^5}{(p^2 - m^2 + i\varepsilon)^3} \\ & + \frac{2}{3} \frac{D_\alpha G_{\alpha\mu}}{(p^2 - m^2 + i\varepsilon)^3} \left[\gamma^\mu (p^2 - m^2) - p^\mu \not{p} \right] - \frac{i}{8} D_\sigma D_\eta G_{\rho\mu} \mathcal{S}_1^{(\tau)\mu\rho\sigma\eta}(p) + \dots, \end{aligned} \quad (1.23)$$

where $\tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}$ denotes the dual gluon field-strength tensor. The last term in Eq. (1.23) contains the genuinely new dimension-seven contribution. For compactness, the corresponding Dirac-Lorentz structure has been collected in $\mathcal{S}_1^{(\tau)\mu\rho\sigma\eta}(p)$, which is given by

$$\begin{aligned} \mathcal{S}_1^{(\tau)\mu\rho\sigma\eta}(p) = & -2 \frac{\not{p} \gamma^\mu}{(p^2 - m^2)^3} (\gamma^\sigma g^{\eta\rho} + \gamma^\rho g^{\eta\sigma} + \gamma^\eta g^{\rho\sigma}) \\ & + 8 \frac{\not{p} \gamma^\mu}{(p^2 - m^2)^4} (\gamma^\sigma p^\eta p^\rho + \gamma^\rho p^\eta p^\sigma + \gamma^\eta p^\rho p^\sigma) \\ & + 8 \frac{2p^\mu \not{p} - \gamma^\mu (p^2 - m^2)}{(p^2 - m^2)^4} (p^\eta g^{\rho\sigma} + p^\rho g^{\eta\sigma} + p^\sigma g^{\eta\rho}) \\ & - 48 \frac{2p^\mu \not{p} - \gamma^\mu (p^2 - m^2)}{(p^2 - m^2)^5} p^\eta p^\rho p^\sigma. \end{aligned} \quad (1.24)$$

Similarly, for the propagator with reversed arguments we write

$$S(0, x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{\mathcal{S}}(p) e^{ip \cdot x}, \quad (1.25)$$

with

$$\tilde{\mathcal{S}}(p) = \tilde{\mathcal{S}}_0(p) + \tilde{\mathcal{S}}_1(p) + \tilde{\mathcal{S}}_2(p) + \dots \quad (1.26)$$

The one-gluon contribution is obtained analogously to Eq. (1.23). It takes the form

$$\begin{aligned} \tilde{\mathcal{S}}_1(p) = & \frac{\tilde{G}_{\sigma\eta}}{(p^2 - m^2 + i\varepsilon)^2} p^\sigma \gamma^\eta \gamma^5 - \frac{2}{3} \frac{p^\alpha D_\alpha G_{\rho\mu}}{(p^2 - m^2 + i\varepsilon)^3} \gamma^\mu p^\rho - 2i \frac{D_\alpha \tilde{G}_{\tau\eta} p^\alpha p^\tau}{(p^2 - m^2 + i\varepsilon)^3} \gamma^\eta \gamma^5 \\ & + \frac{2}{3} \frac{D_\alpha G_{\alpha\mu}}{(p^2 - m^2 + i\varepsilon)^3} \left[\gamma^\mu (p^2 - m^2) - p^\mu \not{p} \right] - \frac{i}{8} D_\sigma D_\eta G_{\rho\mu} \tilde{\mathcal{S}}_1^{(7)\mu\rho\sigma\eta}(p) + \dots \end{aligned} \quad (1.27)$$

As in the case of $\mathcal{S}_1(p)$, the genuinely new dimension-seven contribution has been collected into the Dirac–Lorentz structure $\tilde{\mathcal{S}}_1^{(7)\mu\rho\sigma\eta}(p)$, defined by

$$\begin{aligned} \tilde{\mathcal{S}}_1^{(7)\mu\rho\sigma\eta}(p) = & \frac{2}{(p^2 - m^2)^3} (\gamma^\sigma g^{\eta\rho} + \gamma^\rho g^{\eta\sigma} + \gamma^\eta g^{\rho\sigma}) \not{p} \gamma^\mu \\ & - \frac{8}{(p^2 - m^2)^4} (\gamma^\sigma p^\eta p^\rho + \gamma^\rho p^\eta p^\sigma + \gamma^\eta p^\rho p^\sigma) \not{p} \gamma^\mu \\ & - \frac{8}{(p^2 - m^2)^4} (p^\eta g^{\rho\sigma} + p^\rho g^{\eta\sigma} + p^\sigma g^{\eta\rho}) (2p^\mu \not{p} - \gamma^\mu (p^2 - m^2)) \\ & + \frac{48}{(p^2 - m^2)^5} (2p^\mu \not{p} - \gamma^\mu (p^2 - m^2)) p^\eta p^\rho p^\sigma. \end{aligned} \quad (1.28)$$

The contribution with two external-gluon insertions is obtained from $\mathcal{S}_2(p)$. Retaining only the leading term in the FS expansion of the gauge field, namely the first term in Eq. (1.20), and substituting it into Eq. (1.19) yields the propagator terms containing two field-strength tensors. The corresponding momentum-space expression with finite quark mass is

$$\begin{aligned} \mathcal{S}_2(p) = & - \frac{\not{p} \gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma}{4(p^2 - m^2)^3} G_{\mu\rho} G_{\nu\sigma} - \frac{\gamma^\mu g^{\nu\rho}}{(p^2 - m^2)^3} p^\eta G_{\mu\rho} G_{\nu\eta} - \frac{\gamma^\mu \gamma^\nu \gamma^\rho}{(p^2 - m^2)^3} p^\eta G_{\nu\rho} G_{\mu\eta} \\ & + \frac{1}{4} \left(\frac{2i \gamma^\rho \gamma^5 \epsilon^{\mu\nu\rho\eta} p^\eta - 2 \not{p} g^{\mu\nu}}{(p^2 - m^2)^3} + \frac{4m^2 \gamma^\nu p^\mu}{(p^2 - m^2)^4} \right) G_{\mu\lambda} G_{\nu\lambda} \\ & + \frac{1}{4} \left(\frac{-8i \gamma^\rho \gamma^5 \epsilon^{\mu\nu\rho\eta} p^\eta + 8 \not{p} g^{\mu\nu}}{(p^2 - m^2)^4} + \frac{24m^2 \gamma^\nu p^\mu}{(p^2 - m^2)^5} \right) p^\alpha p^\beta G_{\mu\alpha} G_{\nu\beta} \\ & + \frac{p^2 \gamma^\nu}{(p^2 - m^2)^4} p^\eta G_{\nu\rho} G_{\rho\eta} - \frac{2 \not{p} g^{\nu\rho}}{(p^2 - m^2)^4} p^\alpha p^\beta G_{\nu\alpha} G_{\rho\beta} \\ & + \frac{2 \not{p}}{(p^2 - m^2)^4} p^\alpha p^\beta G_{\sigma\alpha} G_{\sigma\beta}. \end{aligned} \quad (1.29)$$

Similarly, for the propagator with reversed arguments one obtains

$$\begin{aligned}
\tilde{\mathcal{S}}_2(p) = & -\frac{\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu \not{p}}{4(p^2 - m^2)^3} G_{\mu\rho} G_{\nu\sigma} - \frac{\gamma^\nu g^{\mu\rho}}{(p^2 - m^2)^3} p^\eta G_{\nu\rho} G_{\mu\eta} - \frac{\gamma^\rho \gamma^\mu \gamma^\nu}{(p^2 - m^2)^3} p^\eta G_{\mu\rho} G_{\nu\eta} \\
& + \frac{1}{4} \left(\frac{2i \gamma^\rho \gamma^5 \epsilon^{\mu\nu\rho\eta} p^\eta - 2\not{p} g^{\mu\nu}}{(p^2 - m^2)^3} + \frac{4m^2 \gamma^\nu p^\mu}{(p^2 - m^2)^4} \right) G_{\mu\lambda} G_{\nu\lambda} \\
& + \frac{1}{4} \left(\frac{-8i \gamma^\rho \gamma^5 \epsilon^{\mu\nu\rho\eta} p^\eta + 8\not{p} g^{\mu\nu}}{(p^2 - m^2)^4} + \frac{24m^2 \gamma^\nu p^\mu}{(p^2 - m^2)^5} \right) p^\alpha p^\beta G_{\mu\alpha} G_{\nu\beta} \\
& + \frac{p^2 \gamma^\nu}{(p^2 - m^2)^4} p^\eta G_{\nu\rho} G_{\rho\eta} - \frac{2\not{p} g^{\mu\sigma}}{(p^2 - m^2)^4} p^\alpha p^\beta G_{\mu\alpha} G_{\sigma\beta} \\
& + \frac{2\not{p}}{(p^2 - m^2)^4} p^\alpha p^\beta G_{\sigma\alpha} G_{\sigma\beta}.
\end{aligned} \tag{1.30}$$

In the massless limit, $m \rightarrow 0$, the propagator expressions in Eqs. (1.24) and (1.29), restricted throughout to terms containing an odd number of Dirac matrices, reduce to the corresponding results of Ref. [61]. For finite quark masses, our results can be compared with the background-field propagators derived in Refs. [64, 65].

Rusov [64] derived the massive background-field propagator both in coordinate space, where the coefficients are expressed in terms of the modified Bessel functions K_n , and in the corresponding momentum-space representation. In both representations, the gluon field strength is kept in the nonlocal form $G_{\mu\nu}(ux)$, which is suitable for applications to LCSR. For the HQE, however, a local operator expansion about a single spacetime point is required. This is obtained by performing a covariant Taylor expansion of the background field,

$$G_{\mu\nu}(ux) = G_{\mu\nu}(0) + u x^\alpha D_\alpha G_{\mu\nu}(0) + \frac{u^2}{2} x^\alpha x^\beta D_\alpha D_\beta G_{\mu\nu}(0) + \dots, \tag{1.31}$$

followed by replacing the coordinate factors with momentum derivatives acting on the propagator and integrating by parts. Restricting to terms linear in the background gluon field, the resulting local expression is equivalent, after standard tensor and Dirac-algebra identities, to Eq. (1.24), including the terms proportional to $D_\sigma D_\eta G_{\rho\mu}$.

An independent check of the terms containing two field-strength tensors (GG) is provided by Ref. [65], where the massive quark propagator is expanded to multiple insertions of the background gluon field. The corresponding two-gluon contribution agrees, after standard Dirac and tensor identities, with Eq. (1.29). Thus, the DDG and GG contributions to the massive propagator are independently confirmed by Refs. [64, 65], respectively.

The reversed orientation $S(0, x)$ requires separate consideration. Since the FS gauge singles out the origin, translational invariance is not manifest and $S(0, x)$ cannot be obtained from $S(x, 0)$ by a simple translation. As both orientations appear naturally in the HQE, we derive the corresponding local expansion for $S(0, x)$ explicitly (see Eq. (1.25)).

1.4 Light-Cone Sum Rules

In Chapter 4, we study exclusive B -meson decays within the framework of LCSRs. This approach provides a powerful nonperturbative tool for calculating hadronic

matrix elements in processes where both short-distance QCD dynamics and long-distance hadronic effects play an important role, such as exclusive heavy-meson decays, see e.g. [66].

The LCSR framework originated as an extension of the traditional QCD sum rule approach developed by Shifman, Vainshtein, and Zakharov [54, 67] in the late 1970s. In that method, hadronic observables were related to QCD parameters and vacuum condensates through a short-distance OPE. In the 1980s, Balitsky, Braun, and Kolesnichenko [68, 69] reformulated this framework near the light-cone, where vacuum-to-hadron matrix elements of nonlocal quark–gluon operators are expanded in terms of hadronic distribution amplitudes (DAs) instead of vacuum condensates. This light-cone expansion enabled a systematic description of the longitudinal momentum structure of hadrons and established a direct connection between perturbative QCD and exclusive hadronic observables.

Since then, the LCSR approach has become an essential framework for the study of exclusive processes involving heavy hadrons, combining the advantages of the OPE with the flexibility of light-cone factorization. In the following, we describe the LCSR framework in a generic form in order to outline the main conceptual steps and the general workflow of the method before turning to the specific applications considered in this thesis.

The starting point of the practical calculations in the LCSR approach is the analysis of suitably chosen correlation function involving an interpolating current for the hadron of interest and the relevant weak transition current. For a generic B -meson decay into a light hadron, one considers the correlator

$$\Pi_\mu(p, q) = i \int d^4x e^{iq \cdot x} \langle 0 | T \{ j_{\text{light}}(x), J_\mu^{\text{weak}}(0) \} | \bar{B}(p) \rangle, \quad (1.32)$$

where J_μ^{weak} denotes the relevant weak current and j_{light} is an interpolating current that couples to the light hadronic final state. The external B meson in this setup carries momentum p , while q is the momentum transferred through the weak current.

The essential idea of the LCSR method is that the correlation function in Eq. (1.32) admits two complementary representations, each valid in a different kinematical regime. For large spacelike virtualities of the currents, the correlator is dominated by short light-cone distances ($x^2 \approx 0$) and can be expanded using the light-cone OPE. In this representation, the short-distance part of the correlation function is calculable perturbatively in QCD, whereas the long-distance dynamics are encoded in universal nonperturbative light-cone distribution amplitudes (LCDAs). The nonlocal quark–gluon operators appearing in the expansion are systematically organized according to their twist, defined as the difference between the canonical dimension and the Lorentz spin of the operator. Operators with lower twist provide the leading contributions in the large-energy or heavy-mass limit, while higher-twist terms are suppressed by additional powers of the scale. The corresponding LCDAs describe the longitudinal momentum distribution of the partons inside the hadron and provide the essential nonperturbative input of the LCSR framework.

On the other hand, for physical timelike momenta the same correlator admits a hadronic representation. Using the unitarity relation, its imaginary part in the

variable $(p - q)^2$ can be written as

$$\begin{aligned} \frac{1}{\pi} \text{Im} \Pi_\mu(p, q) &= \delta((p - q)^2 - m_H^2) \langle 0 | j_{\text{light}} | H(p - q) \rangle \langle H(p - q) | J_\mu^{\text{weak}} | \bar{B}(p) \rangle \\ &\quad + \rho^{\text{had}}((p - q)^2, q^2), \end{aligned} \quad (1.33)$$

where H denotes the lowest hadronic state carrying the quantum numbers of the interpolating current j_{light} . The second term, described by the spectral density $\rho^{\text{had}}((p - q)^2, q^2)$, contains the contributions of all higher excited hadronic states and multiparticle continuum configurations with the same quantum numbers. In general, the hadronic representation is therefore not composed solely of isolated pole contributions, but also contains continuous spectral distributions associated with broad resonances and multiparticle states.⁴

A fundamental ingredient at the heart of the sum-rule method is the dispersion relation, which follows from the analyticity properties of the correlation function in the variable $(p - q)^2$. Exploiting this analytic structure, the correlator can be represented as

$$\Pi_\mu(p, q) = \frac{1}{\pi} \int ds \frac{\text{Im} \Pi_\mu(s, q^2)}{s - (p - q)^2}. \quad (1.34)$$

As discussed above, the same correlator can be evaluated in two different kinematical regimes. In the physical timelike region, the imaginary part of the correlator is expressed in terms of hadronic intermediate states using the unitarity relation in Eq. (1.33). On the other hand, in the deep Euclidean region, where $(p - q)^2 \ll 0$, the same correlation function is calculated within QCD using the light-cone OPE. Its corresponding spectral density is then obtained from the imaginary part of the OPE result, yielding a perturbative spectral density convoluted with the relevant LCDAs.

At this stage, an important conceptual difference between the two representations becomes apparent. In the hadronic representation, the spectral density is expressed in terms of physical hadronic states, such as mesons, resonances, and multiparticle continuum configurations. In contrast, in the deep Euclidean region, the correlation function is calculated in terms of quark and gluon degrees of freedom using the light-cone OPE, from which the corresponding perturbative spectral density is obtained by taking its imaginary part.

The central assumption allowing one to relate these two descriptions is quark-hadron duality. More specifically, above an effective continuum threshold s_0 , the hadronic spectral density is assumed to be sufficiently well approximated by its perturbative QCD counterpart:

$$\text{Im} \Pi_\mu^{\text{had}}(s, q^2) = \text{Im} \Pi_\mu^{\text{OPE}}(s, q^2), \quad s > s_0. \quad (1.35)$$

Substituting the hadronic spectral representation from Eq. (1.33) into the dispersion relation Eq. (1.34), one obtains the hadronic representation of the correlator,

$$\Pi_\mu^{\text{had}}(p, q) = \frac{\langle 0 | j_{\text{light}} | H(p - q) \rangle \langle H(p - q) | J_\mu^{\text{weak}} | \bar{B}(p) \rangle}{m_H^2 - (p - q)^2} + \int_{s_h}^{\infty} ds \frac{\rho^{\text{had}}(s, q^2)}{s - (p - q)^2}. \quad (1.36)$$

⁴For example, if j_{light} is an interpolating current with the quantum numbers of a pion, the hadronic spectral density contains not only the pion pole, but also excited pion resonances, such as $\pi(1300)$ and $\pi(1800)$, together with multiparticle continuum states (e.g. 3π , $\rho\pi$, etc.) carrying the same quantum numbers.

Here s_h denotes the physical threshold at which the higher-state contribution in the channel of j_{light} starts.⁵

In the Euclidean region, the same correlator is calculated from the light-cone OPE. Its dispersion representation can be written as

$$\Pi_\mu^{\text{OPE}}(p, q) = \frac{1}{\pi} \int_{s_{\text{OPE}}}^{\infty} ds \frac{\text{Im } \Pi_\mu^{\text{OPE}}(s, q^2)}{s - (p - q)^2}, \quad (1.37)$$

where s_{OPE} is the partonic threshold determined by the masses and approximations used in the OPE calculation.⁶

Applying the quark-hadron duality approximation in Eq. (1.35) to the continuum parts of the dispersion relations, one identifies the hadronic spectral density above the effective threshold s_0 with the corresponding OPE spectral density. This allows the higher-state contribution in Eq. (1.36) to be replaced by

$$\int_{s_h}^{\infty} ds \frac{\rho^{\text{had}}(s, q^2)}{s - (p - q)^2} \simeq \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Im } \Pi_\mu^{\text{OPE}}(s, q^2)}{s - (p - q)^2}. \quad (1.38)$$

As discussed above, the same correlation function admits both a hadronic and an OPE description, corresponding to the physical timelike and deep spacelike regions, respectively. Writing the correlator in the form of the dispersion relation in Eq. (1.34) allows one to connect both kinematical domains through the analyticity of the correlation function in the variable $s = (p - q)^2$. While the hadronic and OPE calculations are each valid only in their corresponding kinematical regions, the dispersion relation itself represents the analytic continuation of the same correlator and is therefore valid throughout the complex s -plane up to the physical singularities. In this way, the hadronic and OPE representations correspond to the same underlying analytic function and can therefore be matched:

$$\Pi_\mu^{\text{had}}((p - q)^2, q^2) = \Pi_\mu^{\text{OPE}}((p - q)^2, q^2), \quad (1.39)$$

Using the explicit hadronic and OPE representations in Eqs. (1.36) and (1.37), one obtains

$$\begin{aligned} \frac{1}{\pi} \int_{s_{\text{OPE}}}^{\infty} ds \frac{\text{Im } \Pi_\mu^{\text{OPE}}(s, q^2)}{s - (p - q)^2} &= \frac{\langle 0 | j_{\text{light}} | H(p - q) \rangle \langle H(p - q) | J_\mu^{\text{weak}} | \bar{B}(p) \rangle}{m_H^2 - (p - q)^2} \\ &+ \int_{s_h}^{\infty} ds \frac{\rho^{\text{had}}(s, q^2)}{s - (p - q)^2}. \end{aligned} \quad (1.40)$$

⁵For example, if j_{light} carries the quantum numbers of the pion, the lowest multiparticle continuum starts at the three-pion threshold, $s_h = (3m_\pi)^2 = 9m_\pi^2$.

⁶If j_{light} carries the π^+ quantum numbers, the perturbative spectral density corresponds to the production threshold of the underlying $u\bar{d}$ quark pair, $s_{\text{OPE}} = (m_u + m_d)^2$. In the massless light-quark limit, this reduces to $s_{\text{OPE}} = 0$.

Applying the quark–hadron duality approximation in Eq. (1.38) and subtracting the continuum contribution from both sides of Eq. (1.40) isolates the ground-state pole contribution. This yields

$$\frac{1}{\pi} \int_{s_{\text{OPE}}}^{s_0} ds \frac{\text{Im } \Pi_{\mu}^{\text{OPE}}(s, q^2)}{s - (p - q)^2} = \frac{\langle 0 | j_{\text{light}} | H(p - q) \rangle \langle H(p - q) | J_{\mu}^{\text{weak}} | \bar{B}(p) \rangle}{m_H^2 - (p - q)^2}. \quad (1.41)$$

It is important to emphasize that the duality relation in Eq. (1.38) is weaker than the local duality relation introduced in Eq. (1.35). The latter corresponds to a pointwise equality between the hadronic and OPE spectral densities at each value of s above the effective threshold s_0 . In practice, however, such a local equality is not expected to hold because of the resonance structure of the physical hadronic spectrum. Instead, the duality assumption employed in QCD sum rules is understood in a semi-local (or averaged) sense: after integration over a sufficiently broad energy interval, the hadronic spectral density is approximated by its perturbative QCD counterpart. It is this semi-local duality that underlies the continuum subtraction in Eq. (1.38).

At this stage, the continuum contribution has been subtracted, but the relation in Eq. (1.41) still contains possible subtraction terms from the dispersion relation and depends explicitly on the external variable $(p - q)^2$. To remove these subtraction terms, improve the convergence of the OPE, and suppress the sensitivity to the region close to the continuum threshold, one performs a Borel transformation with respect to $(p - q)^2$.

The Borel transformation is defined by its action on the dispersion kernel,

$$\mathcal{B}_{M^2} \left[\frac{1}{s - (p - q)^2} \right] = \frac{1}{M^2} e^{-s/M^2}, \quad (1.42)$$

where M^2 is the Borel parameter. Applying this transformation to both sides of Eq. (1.41) gives

$$\frac{1}{\pi} \int_{s_{\text{OPE}}}^{s_0} ds e^{-s/M^2} \text{Im } \Pi_{\mu}^{\text{OPE}}(s, q^2) = e^{-m_H^2/M^2} \langle 0 | j_{\text{light}} | H(p - q) \rangle \langle H(p - q) | J_{\mu}^{\text{weak}} | \bar{B}(p) \rangle. \quad (1.43)$$

Eq. (1.43) represents the generic structure of a LCSR. The left-hand side contains the OPE representation of the correlation function, calculated in QCD in terms of perturbative kernels and LCDAs. The right-hand side is expressed through hadronic quantities associated with the lowest physical state, such as decay constants and form factors. In this way, the sum rule connects the quark–gluon description of QCD with observable hadronic parameters.⁷

To summarize, the LCSR method starts from a suitably chosen correlation function involving the weak transition current and an interpolating current for the hadronic

⁷The name “sum rule” originates from the unitarity relation in Eq. (1.33), where the correlation function is expressed as a sum over hadronic states and their corresponding spectral densities. After applying quark–hadron duality and the Borel transformation, one obtains a relation (or “rule”) that allows this spectral sum to be evaluated in terms of QCD quantities.

state of interest. Using the unitarity relation, the correlator is expressed in terms of hadronic states, thereby relating it to the desired hadronic matrix element. The same correlation function is then calculated in the deep Euclidean region using the light-cone OPE within QCD. Assuming quark–hadron duality, the contributions of higher hadronic states and continuum configurations are approximated by the corresponding OPE spectral density above an effective threshold. Finally, a Borel transformation is applied to suppress the higher-state contributions and improve the convergence of the expansion. The resulting sum rule relates the perturbatively calculable OPE expression to the desired hadronic parameters, such as decay constants and form factors.

The main theoretical uncertainties in the LCSR approach typically originate from the choice of the continuum threshold s_0 , the Borel parameter M^2 , and the nonperturbative input entering the LCDAs. The continuum threshold and Borel window are not arbitrary, but are chosen according to the standard criteria of QCD sum rules. In practice, the continuum threshold s_0 is typically determined from the corresponding two-point sum rule and reflects the onset of excited hadronic states and the continuum. The Borel parameter is then chosen such that the light-cone OPE remains convergent while the ground-state contribution is sufficiently enhanced relative to higher-state and continuum contributions. Residual variations with respect to s_0 and the Borel parameter are included in the uncertainty estimate of the final results.

Having introduced the LCSR approach, it is instructive to briefly compare it with another major first-principles non-perturbative method, namely lattice QCD, originally introduced by Wilson [70]. In this framework, spacetime is discretized into a finite Euclidean lattice, allowing hadronic observables to be computed numerically from first principles; for a recent review, see [71]. In contrast, LCSR is based on the light-cone OPE and combines perturbatively calculable coefficient functions with universal LCDAs. Although lattice QCD is systematically improvable and, for many observables, provides the most precise theoretical predictions, it is computationally demanding, requiring large-scale numerical simulations on high-performance computing facilities. Consequently, the calculation of a new hadronic quantity or the inclusion of additional operator structures often represents a substantial computational effort. By comparison, LCSR provides analytic expressions that can be adapted more readily to new processes and BSM operator structures. Moreover, in the context of hadronic form factors, lattice QCD and LCSR are applicable in complementary kinematic regimes. Lattice QCD provides the most precise results at low hadronic recoil (large momentum transfer, large q^2), whereas LCSR is naturally suited to the large-recoil region (small momentum transfer, small q^2), where the light-cone expansion converges most effectively. Although lattice QCD is applicable to a much broader class of non-perturbative observables, including hadron masses, decay constants, bag parameters, and matrix elements of local operators, the comparison here is restricted to form-factor calculations. In this context, LCSR and lattice QCD should be regarded as complementary rather than competing approaches. Their predictions are frequently combined to obtain reliable form-factor determinations over the full kinematic range, while providing valuable cross-checks in the region where both methods are applicable.

The explicit form of the correlation functions, the choice of interpolating currents, and the relevant distribution amplitudes employed in this work will be introduced and discussed in detail in [Chapter 4](#), where the general LCSR framework described above will be applied to the study of the $B \rightarrow p$ transition form factors.

Chapter 2

SM Dimension-Seven Two-Quark Contributions

“I do not like that we get these new matrix elements.”

“Then complain.”

“To whom?”

“To God. Why did you make nature this way?”

— A Discussion with Alexey Pivovarov

In this chapter, the complete two-quark contributions to the dimension-seven term $\Gamma_7^{(0)}$ in the HQE are derived at leading order in QCD, corresponding to corrections of order $\mathcal{O}(1/m_b^4)$ to the inclusive non-leptonic decay width. These contributions constitute the $1/m_b$ corrections to the dimension-six two-quark Darwin term Γ_6 . The analysis is restricted to the CKM-favoured transitions $b \rightarrow c\bar{u}d$ and $b \rightarrow c\bar{c}s$, which provide the dominant contributions to the total decay width of B mesons. The calculation is performed within the framework introduced in [Chapter 1](#), and the complete analytical matching coefficients for the dimension-seven two-quark operator basis are presented.

The chapter begins with a discussion of the HQE operator basis and the matching procedure, followed by the derivation of the corresponding Wilson coefficients. The resulting analytical expressions constitute the first complete determination of the dimension-seven matching coefficients for the CKM-favoured non-leptonic channels and provide an important ingredient for future precision studies of heavy-hadron lifetimes and related observables.

2.1 Outline of the Calculation

The starting point of the calculation is the transition operator entering [Eq. \(1.13\)](#),

$$\mathcal{T} = \text{Im } i \int d^4x T\{\mathcal{H}_{\text{eff}}(x)\mathcal{H}_{\text{eff}}(0)\}, \quad (2.1)$$

whose forward matrix element determines the inclusive decay width within the HQE. The effective weak Hamiltonian is given by the SM Hamiltonian in [Eq. \(1.9\)](#). The calculation presented in this chapter is restricted to the current–current sector of the SM effective Hamiltonian for the generic non-leptonic transition $b \rightarrow q_1\bar{q}_2q_3$. Accordingly, only the operators Q_1 and Q_2 are considered. For completeness, they are given by

$$Q_1^{q_1q_2q_3} = (\bar{q}_1^i\gamma_\mu P_L b^i)(\bar{q}_3^j\gamma^\mu P_L q_2^j), \quad Q_2^{q_1q_2q_3} = (\bar{q}_1^i\gamma_\mu P_L b^j)(\bar{q}_3^j\gamma^\mu P_L q_2^i). \quad (2.2)$$

Throughout this chapter, the operators are evaluated for the $b \rightarrow c\bar{u}d$ and $b \rightarrow c\bar{c}s$ transitions.

The transition operator can be decomposed according to the insertion of the current–current operators as

$$\mathcal{T}^{(q_1\bar{q}_2q_3)} = C_1^2 \mathcal{T}_{11}^{(q_1\bar{q}_2q_3)} + 2C_1C_2 \mathcal{T}_{12}^{(q_1\bar{q}_2q_3)} + C_2^2 \mathcal{T}_{22}^{(q_1\bar{q}_2q_3)}, \quad (2.3)$$

where $\mathcal{T}_{ij}^{(q_1\bar{q}_2q_3)}$ denotes the contribution originating from the insertion of the operator pair $Q_i^{q_1\bar{q}_2q_3}$ and $Q_j^{q_1\bar{q}_2q_3}$.

Substituting the current–current operators into the time-ordered product in Eq. (1.13) and performing the Wick contractions yields the following expressions for the two-quark contributions:

$$\begin{aligned} \mathcal{T}_{11}^{(2q)} = & -\frac{G_F^2}{2} |V_{q_1b}|^2 |V_{q_3q_2}|^2 i \int d^4x \bar{b}^j \gamma_\nu (1 - \gamma_5) iS_{jk}^{(q_1)}(0, x) \gamma_\mu (1 - \gamma_5) b^k(x) \\ & \times \text{Tr} \left[\gamma^\nu (1 - \gamma_5) iS_{lm}^{(q_3)}(0, x) \gamma^\mu (1 - \gamma_5) iS_{ml}^{(q_2)}(x, 0) \right] + (x \leftrightarrow 0), \end{aligned} \quad (2.4)$$

$$\begin{aligned} \mathcal{T}_{12}^{(2q)} = & -\frac{G_F^2}{2} |V_{q_1b}|^2 |V_{q_3q_2}|^2 i \int d^4x \bar{b}^j \gamma_\nu (1 - \gamma_5) iS_{lk}^{(q_1)}(0, x) \gamma_\mu (1 - \gamma_5) b^k(x) \\ & \times \text{Tr} \left[\gamma^\nu (1 - \gamma_5) iS_{jr}^{(q_3)}(0, x) \gamma^\mu (1 - \gamma_5) iS_{rl}^{(q_2)}(x, 0) \right] + (x \leftrightarrow 0), \end{aligned} \quad (2.5)$$

$$\begin{aligned} \mathcal{T}_{22}^{(2q)} = & -\frac{G_F^2}{2} |V_{q_1b}|^2 |V_{q_3q_2}|^2 i \int d^4x \bar{b}^j \gamma_\nu (1 - \gamma_5) iS_{lm}^{(q_1)}(0, x) \gamma_\mu (1 - \gamma_5) b^k(x) \\ & \times \text{Tr} \left[\gamma^\nu (1 - \gamma_5) iS_{jk}^{(q_3)}(0, x) \gamma^\mu (1 - \gamma_5) iS_{lm}^{(q_2)}(x, 0) \right] + (x \leftrightarrow 0), \end{aligned} \quad (2.6)$$

The overall minus sign in Eqs. (2.4) to (2.6) originates from the closed light-quark fermion loop. The notation $(x \leftrightarrow 0)$ denotes the contribution obtained by interchanging the spacetime coordinates x and 0, corresponding to the second ordering of the effective Hamiltonians in the time-ordered product.

To obtain the two-quark contributions at dimension seven, the matching must be performed up to operators containing four covariant derivatives acting on the heavy-quark fields, schematically of the form

$$\bar{b} iD_\mu iD_\nu iD_\rho iD_\sigma b.$$

Using the commutation relations of the covariant derivatives, these operators can be expressed in terms of structures containing either two field-strength tensors,

$$G_{\mu\nu}G_{\rho\sigma}, \quad G_{\mu\nu} = i[iD_\mu, iD_\nu],$$

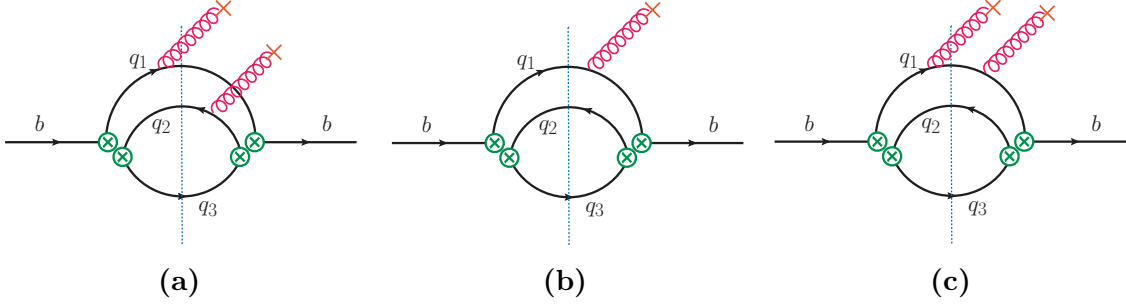


Figure 2.1: Representative diagrams for dimension-7 two-quark contributions to $b \rightarrow q_1 \bar{q}_2 q_3$ decays: (a) two gluons attached to different quark lines, (b) a single gluon with two covariant derivatives $D_\mu D_\nu G_{\rho\sigma}$, (c) two gluons attached to the same propagator. Shown here are representative topologies; the gluons can be attached to different quark propagators depending on the colour structure (see text). Dashed blue lines indicate the cut for forward matrix elements.

or a single field-strength tensor accompanied by two additional covariant derivatives,

$$D_\mu D_\nu G_{\rho\sigma} = i[iD_\mu, [iD_\nu, [iD_\rho, iD_\sigma]]].$$

Such structures are generated by expanding the quark propagators in the external gluon field (see [Section 1.3](#)). At dimension seven, three distinct classes of propagator expansions contribute:

1. **Different-propagator insertion (Fig. 2.1a):** One background gluon is emitted from each of two different propagators, giving rise to operators containing two field-strength tensors ($G_{\mu\nu}G_{\rho\sigma}$). These operators are generated by combining the first two terms of [Eq. \(1.23\)](#) (or [Eq. \(1.27\)](#)), one from each propagator.
2. **Single-gluon insertion (Fig. 2.1b):** A single background gluon is emitted from one propagator. The relevant terms arise from the $D_\mu D_\nu G_{\rho\sigma}$ structures in the one-gluon propagator expansion, see [Eqs. \(1.23\)](#), [\(1.24\)](#) and [\(1.27\)](#).
3. **Double-gluon insertion (Fig. 2.1c):** Two background gluons are emitted from the same propagator through the double-gluon term in the propagator expansion. The relevant $G_{\mu\nu}G_{\rho\sigma}$ structures are contained in [Eqs. \(1.29\)](#) and [\(1.30\)](#).

Before proceeding with the explicit evaluation, we first analyse the colour structure of the three classes of propagator expansions. The colour algebra determines which external-gluon emissions give rise to non-vanishing contributions and which diagrams vanish identically. Consequently, only the allowed colour structures need to be considered in the matching calculation. The momentum-space propagators introduced above can be decomposed according to the number of background-gluon insertions as

$$\mathcal{S}_{jk}^{(q_i)}(p) = \mathcal{S}_0^{(q_i)}(p) \delta_{jk} + \mathcal{S}_1^{(q_i)a}(p) t_{jk}^a + \mathcal{S}_2^{(q_i)ab}(p) (t^a t^b)_{jk} + \dots, \quad (2.7)$$

Contribution	Different propagators (GG)	Single insertion (DDG)	Double insertion (GG)
$\mathcal{T}_{11} (Q_1 \otimes Q_1)$	(q_2, q_3)	q_1	q_1, q_2, q_3
$\mathcal{T}_{22} (Q_2 \otimes Q_2)$	(q_1, q_2)	q_3	q_1, q_2, q_3
$\mathcal{T}_{12} (Q_1 \otimes Q_2)$	$(q_1, q_2), (q_1, q_3), (q_2, q_3)$	q_1, q_2, q_3	q_1, q_2, q_3

Table 2.1: Allowed gluon-emission topologies for the different operator products. The ordered pairs indicate the propagators from which the two gluons are emitted.

where $i = 1, 2, 3$ labels the quark flavour. The quantities $\mathcal{S}_0^{(q_i)}$, $\mathcal{S}_1^{(q_i)}$, and $\mathcal{S}_2^{(q_i)}$ denote the propagator contributions (see Eqs. (1.16), (1.21), (1.22), (1.25) and (1.26)), while the explicit colour matrices have been factored out. To illustrate the role of the colour decomposition, consider the contribution $\mathcal{T}_{11}^{(q_1 \bar{q}_2 q_3)}$ in Eq. (2.4). The allowed propagator expansions are determined entirely by the associated colour contractions:

- **Different-propagator insertion.** When one background gluon is emitted from each of the q_1 and q_2 propagators, the colour structure becomes

$$S_{jk}^{(q_1)} S_{lm}^{(q_3)} S_{ml}^{(q_2)} \longrightarrow t_{jk}^a \delta_{lm} t_{ml}^b = t_{jk}^a \text{Tr}(t^b) = 0, \quad (2.8)$$

and therefore this topology does not contribute.

- **Single-gluon insertion.** If the background gluon is emitted from the q_1 propagator while the remaining propagators are taken at leading order, one finds

$$S_{jk}^{(q_1)} S_{lm}^{(q_3)} S_{ml}^{(q_2)} \longrightarrow t_{jk}^a \delta_{lm} \delta_{ml} = N_c t_{jk}^a, \quad (2.9)$$

which is non-vanishing.

- **Double-gluon insertion.** If both background gluons are emitted from the q_1 propagator, the colour structure becomes

$$S_{jk}^{(q_1)} S_{lm}^{(q_3)} S_{ml}^{(q_2)} \longrightarrow (t^a t^b)_{jk} \delta_{lm} \delta_{ml} = N_c (t^a t^b)_{jk}, \quad (2.10)$$

which is likewise non-vanishing.

Applying the same procedure to the remaining gluon-emission topologies and the other operator products yields the allowed gluon-emission patterns summarized in Table 2.1.

The calculation of the dimension-seven matching coefficients follows the same general strategy as the determination of the Darwin-term contribution at dimension six presented in Ref. [62]. The individual steps can be summarized as follows:

1. **Insertion of the propagator expansion.** The appropriate terms of the external-field propagator discussed in Section 1.3 are substituted into the transition operators in Eqs. (2.4) to (2.6), according to the gluon-emission topology under consideration.

2. **Coordinate-space integrations.** The heavy-quark field is written as

$$b(x) = e^{-ip_b \cdot x} b(0), \quad (2.11)$$

where p_b denotes the momentum of the external b quark.¹ The subsequent x -integration generates momentum-conserving δ -functions, allowing the loop momenta to be introduced.

3. **Loop-integral reduction.** After performing the coordinate-space integration and using the resulting δ -functions to eliminate the constrained momenta, the remaining amplitudes can be expressed in terms of scalar loop integrals of the generic form

$$\mathcal{I}_{n_1 n_2 n_3} = \int \frac{d^4 l_1}{(2\pi)^4} \int \frac{d^4 l_2}{(2\pi)^4} \frac{f(p_b, l_1, l_2)}{[l_1^2 - m_1^2]^{n_1} [l_2^2 - m_2^2]^{n_2} [(l_1 + l_2 - p_b)^2 - m_3^2]^{n_3}}, \quad (2.12)$$

where $n_i \in \mathbb{N}_0$ and $f(p_b, l_1, l_2)$ denotes a polynomial in the independent scalar products. The Feynman prescription $+i\epsilon$ is understood for all propagator denominators and is suppressed throughout the following.

4. **IBP reduction.** Since the integrals are UV finite, the calculation is performed directly in four dimensions. The loop integrals are reduced to a small set of master integrals using integration-by-parts (IBP) identities [72, 73]. The corresponding master integrals are defined in Section B.2; in particular, the sunrise master integrals are introduced in Eqs. (B.5) and (B.6). Their imaginary parts are known analytically [74] (see also Refs. [62, 63]). The corresponding master integrals admit compact analytical expressions whenever at most two internal masses are non-vanishing. For the general case with three massive propagators, however, the analytic structure becomes considerably more involved and is known to involve elliptic functions [75–78]. In the present calculation, finite light-quark masses are introduced where required and are taken to zero only after the corresponding expressions have been combined. Consequently, all master integrals can be expressed analytically in terms of the known results with at most two massive propagators, with the exception of the $b \rightarrow c\bar{c}s$ transition. In this case, all three internal masses are non-vanishing, such that the individual master integrals are not available in closed analytical form. As discussed in Section B.2, the relevant combination entering the physical matching coefficients becomes analytic only after combining the two-quark contributions with the corresponding gluon corrections to the four-quark operators.

5. **Matching onto the HQE operator basis.** After evaluating the loop integrals and taking the imaginary part, the resulting amplitudes assume the generic form

$$\mathcal{G}(p_b) \langle B(p_B) | \bar{b}_v(0) \mathcal{M}^{\mu_1 \dots \mu_4} (iD_{\mu_1}) \dots (iD_{\mu_4}) b_v(0) | B(p_B) \rangle, \quad (2.13)$$

¹This follows from translation invariance of the free field and reflects that the external b quark is described by a plane-wave state with momentum p_b .

where $\mathcal{G}(p_b)$ contains the kinematic dependence on the internal quark masses and the external b -quark momentum p_b , while $\mathcal{M}^{\mu_1 \dots \mu_4}$ denotes the corresponding Dirac-Lorentz structure. The field b_v is the HQET heavy-quark field introduced in Eq. (B.2). At order $\mathcal{O}(1/m_b^4)$, operators containing up to four covariant derivatives are retained. Furthermore, employing the decomposition

$$p_b^\mu = m_b v^\mu + iD^\mu, \quad (2.14)$$

lower-dimensional operators generate additional contributions at $\mathcal{O}(1/m_b^4)$ through the expansion in powers of the covariant derivative. A detailed discussion of this procedure can be found in Appendix A of Ref. [62]. Finally, the resulting matrix elements are projected onto the HQE operator basis collected in Eqs. (B.1), (B.3) and (B.4) using the trace formulas of Ref. [79]. In the non-leptonic case, additional colour-singlet matrix elements arise, requiring the supplementary trace relations derived in Section B.3.

Following the procedure outlined above, the explicit evaluation reveals several general features of the dimension-seven matching. For the contributions $\mathcal{T}_{11}^{(q_1 \bar{q}_2 q_3)}$ and $\mathcal{T}_{22}^{(q_1 \bar{q}_2 q_3)}$, all different-propagator (GG) topologies and all single-gluon (DDG) emissions vanish identically. For the mixed contribution $\mathcal{T}_{12}^{(q_1 \bar{q}_2 q_3)}$, only a subset of the possible topologies survives: among the different-propagator emissions, only the (q_1, q_3) configuration gives a non-vanishing result, while for the single-gluon topology only emission from the q_2 propagator contributes. In contrast, the double-gluon (GG) topology contributes for all propagator attachments in $\mathcal{T}_{11}^{(q_1 \bar{q}_2 q_3)}$, $\mathcal{T}_{12}^{(q_1 \bar{q}_2 q_3)}$, and $\mathcal{T}_{22}^{(q_1 \bar{q}_2 q_3)}$. These cancellations are not a consequence of the colour structure alone, but arise from the interplay between the Dirac algebra and the antisymmetry of the gluon field-strength tensor. Similar patterns were already observed in the dimension-five ($\mathcal{O}(1/m_b^2)$) HQE calculation in coordinate space; see, for example, Ref. [62]. Furthermore, although the FS gauge explicitly breaks translational invariance, such that $S(x, 0) \neq S(0, x)$ (see Section 1.3), the contributions involving $S(x, 0)$ and $S(0, x)$ always combine to produce an overall factor of two in the decay rate. This behaviour is identical to that observed in previous tree-level HQE calculations at lower dimensions.

2.1.1 Mixing with Four-Quark Operators

Soft gluon emissions from massless light-quark propagators give rise to infrared (IR) divergences in the matching calculation. In the present work, these singularities are regulated by retaining finite light-quark masses. Consequently, the IR divergences are represented by logarithmic terms of the form $\log(m_q/m_b)$, where m_q serves solely as an IR regulator. These IR divergences cancel through the mixing of the gluon corrections to the dimension-seven four-quark operators, illustrated in Figure 2.2, into the dimension-seven two-quark operators. Consequently, the resulting matching coefficients are IR finite.

In this work, we adopt the conventional normalized operator basis introduced in Ref. [80] to parameterize the matrix elements of the dimension-seven four-quark

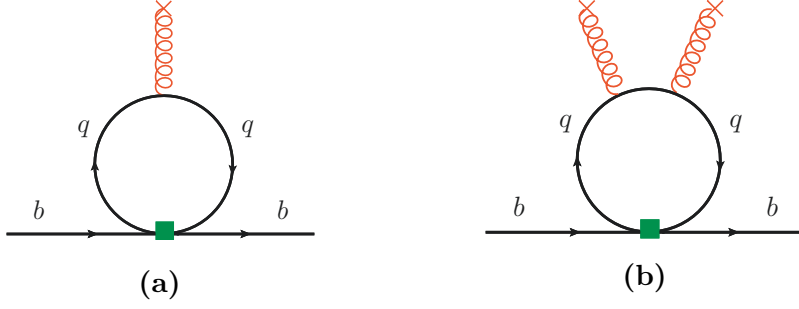


Figure 2.2: Diagrams for the mixing between two-quark and four-quark dimension-seven operators in the HQE. (a) Single gluon emission from a light quark propagator, corresponding to the D^2G term. (b) Double gluon emission from a light quark propagator, corresponding to the G^2 term.

HQE operators. The corresponding operators are

$$\begin{aligned}
R_0^q &= \frac{2}{m_b^2} \bar{b}_i \gamma^\mu \gamma_5 \vec{D}^\alpha b_i \bar{q}_j \gamma_\mu P_L \vec{D}_\alpha q_j, & R_1^q &= \frac{4}{m_b^2} \bar{b}_i \gamma^\mu P_L \vec{D}^\alpha b_i \bar{q}_j \gamma_\mu P_L \vec{D}_\alpha q_j, \\
R_2^q &= \frac{4}{m_b^2} \bar{b}_i \gamma^\mu P_L \vec{D}^\nu b_i \bar{q}_j \gamma_\nu P_L \vec{D}_\mu q_j, & R_3^q &= \frac{4m_q}{m_b} \bar{b}_i P_L b_i \bar{q}_j P_L q_j,
\end{aligned} \tag{2.15}$$

where $q = u, d, s$. The corresponding colour-rearranged operators, denoted by $\tilde{R}_i^q$, are obtained from R_i^q by interchanging the colour indices of the b - and q -quark fields.

To evaluate the mixing diagrams shown in Figure 2.2, we consider the one-loop matrix elements of the four-quark operators in Eq. (2.15) in the presence of a soft background gluon field $A_\mu(x)$. These are obtained by computing the time-ordered product of the operators R_i^q (and analogously $\tilde{R}_i^q$) with the interaction part of the QCD Lagrangian in Eq. (A.7),

$$\mathcal{L}_{\text{QCD}}^{\text{int}}(x) = \sum_q \bar{q}(x) \not{A}(x) q(x), \tag{2.16}$$

which yields

$$\begin{aligned}
\langle R_i^q \rangle_{1\text{-loop}} &= \left\langle i \int d^4z \text{T} \{ R_i^q, \mathcal{L}_{\text{QCD}}^{\text{int}}(z) \} \right\rangle \\
&\quad + \frac{i^2}{2} \left\langle \int d^4z d^4z' \text{T} \{ R_i^q, \mathcal{L}_{\text{QCD}}^{\text{int}}(z) \mathcal{L}_{\text{QCD}}^{\text{int}}(z') \} \right\rangle + \dots
\end{aligned} \tag{2.17}$$

The first term generates the single-gluon diagrams shown in Figure 2.2a, whereas the second term gives rise to the corresponding double-gluon diagrams in Figure 2.2b. The ellipsis denotes higher-order insertions of the interaction Lagrangian involving three or more gluon fields, which do not contribute at the order considered in this work.

The loop integrals generated by both the single- and double-gluon insertions contain UV as well as IR divergences. The UV divergences are regularized using dimensional regularisation, working in $D = 4 - 2\epsilon$ dimensions. Throughout the

calculation, the Dirac algebra is performed in the naive dimensional regularisation (NDR) scheme, satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad g_{\mu\nu}g^{\mu\nu} = D, \quad \{\gamma_\mu, \gamma_5\} = 0. \quad (2.18)$$

To ensure a consistent cancellation of the IR singularities with the corresponding two-quark contributions, the IR divergences are regularized by retaining finite masses for the light quarks, following the same prescription adopted in the two-quark matching. Consequently, the intermediate expressions contain logarithms of the form $\log(m_q/m_b)$, which cancel in the final matching coefficients.

Before evaluating the one-loop matrix elements, it is convenient to rewrite the heavy-quark field in terms of the HQET field according to

$$b(x) = e^{-im_b v \cdot x} b_v(x). \quad (2.19)$$

Hence, a covariant derivative acting on the heavy-quark field becomes

$$iD_\alpha b(x) = e^{-im_b v \cdot x} (m_b v_\alpha + iD_\alpha) b_v(x). \quad (2.20)$$

Moreover, in the FS gauge, the gauge potential vanishes at the origin, $A_\mu(0) = 0$ (see e.g. Eq. (1.20)). Since all local operators are evaluated at the expansion point $x = 0$, the covariant derivative acting on the heavy-quark field reduces to

$$iD_\alpha b(0) = (m_b v_\alpha + i\partial_\alpha) b_v(0). \quad (2.21)$$

This decomposition is applied throughout the evaluation of the dimension-seven four-quark operator matrix elements. At the order considered in this work, the derivative acting on the HQET field generates higher-order corrections and can therefore be neglected, such that

$$D_\alpha b(0) \rightarrow -i m_b v_\alpha b_v(0). \quad (2.22)$$

Using the decomposition in Eq. (2.22) considerably simplifies the heavy-quark structures appearing in the operators R_i^q . A further simplification is achieved by performing a Fierz transformation of the four-quark operators. This avoids the appearance of Dirac traces containing γ_5 , allowing the Dirac algebra to be carried out entirely within the NDR scheme. Only the operators R_0^q and R_1^q are Fierz transformed. The operators R_2^q and R_3^q are treated in their original form. Since the Dirac indices in these operators are not part of the closed Dirac traces arising in the loop calculation, the corresponding Dirac algebra is identical in four and $D = 4 - 2\epsilon$ dimensions. A Fierz transformation is therefore unnecessary. The resulting operators read

$$R_0^q = \frac{2i v^\alpha}{m_b} \bar{b}_{v,i} \gamma_\mu P_L D_\alpha q_j \bar{q}_j \gamma^\mu P_L b_{v,i} + \frac{4i v^\alpha}{m_b} \bar{b}_{v,i} P_R D_\alpha q_j \bar{q}_j P_L b_{v,i} + \mathcal{O}\left(\frac{1}{m_b^2}\right), \quad (2.23)$$

$$R_1^q = -\frac{4i v^\alpha}{m_b} \bar{b}_{v,i} \gamma_\mu P_L D_\alpha q_j \bar{q}_j \gamma^\mu P_L b_{v,i} + \mathcal{O}\left(\frac{1}{m_b^2}\right), \quad (2.24)$$

$$R_2^q = -\frac{4i v^\nu}{m_b} \bar{b}_{v,i} \gamma^\mu P_L b_{v,i} \bar{q}_j \gamma_\nu P_L D_\mu q_j + \mathcal{O}\left(\frac{1}{m_b^2}\right), \quad (2.25)$$

$$R_3^q = 4 \frac{m_q}{m_b} \bar{b}_{v,i} P_L b_{v,i} \bar{q}_j P_L q_j. \quad (2.26)$$

The $\mathcal{O}(1/m_b^2)$ terms shown for $R_{0,1,2}$ originate from neglecting the residual derivative acting on the HQET field in Eq. (2.20). No analogous term appears for R_3 , since this operator contains no derivative acting on the heavy-quark field.

To illustrate the calculation of the mixing diagrams, we consider the single-gluon contribution generated by the colour-rearranged operator $\tilde{R}_1^q$. The remaining operators and topologies are treated analogously.

$$\left\langle \tilde{R}_1^q \right\rangle_{1\text{-loop}} = \left\langle i \int d^4 z \, \text{T} \left\{ \frac{-4i v^\alpha}{m_b} \bar{b}_{v,i} \gamma_\mu P_L D_\alpha q_i \bar{q}_j \gamma^\mu P_L b_{v,j}, \bar{q}_l(z) \mathcal{A}_{lm}(z) q_m(z) \right\} \right\rangle. \quad (2.27)$$

Contracting the light-quark fields gives

$$\left\langle \tilde{R}_1^q \right\rangle_{1\text{-loop}} = \frac{4v^\alpha}{m_b} \int d^4 z \left\langle \bar{b}_{v,i} \gamma_\mu P_L \left[D_\alpha S_{il}^{(q)}(0, z) \right] \mathcal{A}_{lm}(z) S_{mj}^{(q)}(z, 0) \gamma^\mu P_L b_{v,j} \right\rangle. \quad (2.28)$$

Using the FS gauge, where $A_\alpha(0) = 0$, the covariant derivative acting on the endpoint of the propagator reduces to an ordinary derivative,

$$D_\alpha S^{(q)}(0, z) = \partial_\alpha S^{(q)}(x, z) \Big|_{x=0}. \quad (2.29)$$

Using the momentum-space representation of the propagator, this becomes

$$D_\alpha S^{(q)}(0, z) = \int \frac{d^D p}{(2\pi)^D} (-ip_\alpha) \mathcal{S}^{(q)}(p) e^{ip \cdot z}. \quad (2.30)$$

Therefore, at the level of the momentum-space integrand, the action of the derivative amounts to inserting a factor $-ip_\alpha$ on the corresponding propagator line,

$$\left[D_\alpha S_{il}^{(q)}(0, z) \right] \longrightarrow -ip_\alpha S_{il}^{(q)}(0, z). \quad (2.31)$$

Using this replacement in Eq. (2.28), one obtains

$$\left\langle \tilde{R}_1^q \right\rangle_{1\text{-loop}} = - \frac{4i v^\alpha}{m_b} \int d^4 z \left\langle \bar{b}_{v,i} \gamma_\mu P_L p_\alpha S_{il}^{(q)}(0, z) \mathcal{A}_{lm}(z) S_{mj}^{(q)}(z, 0) \gamma^\mu P_L b_{v,j} \right\rangle. \quad (2.32)$$

We recognize the combination

$$\int d^4 z \, iS_0^{(q)}(0, z) i\mathcal{A}(z) iS_0^{(q)}(z, 0) = \lim_{x \rightarrow 0} iS_1^{(q)}(x, 0), \quad (2.33)$$

as the limit of the one-gluon contribution to the quark propagator, cf. Eq. (1.18). In the FS gauge, the latter admits the Fourier representation

$$\lim_{x \rightarrow 0} iS_1^{(q)}(x, 0) = \int \frac{d^D l}{(2\pi)^D} i\mathcal{S}_1^{(q)}(l), \quad (2.34)$$

where $\mathcal{S}_1^{(q)}(l)$ is given in Eq. (1.23). Inserting the explicit propagator expression in Eq. (2.34), evaluating the resulting loop integrals using the scalar tadpole formula of Eq. (B.14), and simplifying the Dirac algebra in dimensional regularisation, the matrix element is reduced to local two-quark operators. Finally, using the trace formulas of Ref. [79] to express these operators in terms of the HQE basis (s_B^4 , s_{qB}^4 , r_E^4 , r_G^4 , ...) defined in Eqs. (B.1), (B.3) and (B.4), one obtains

$$\langle \tilde{R}_1^q \rangle_{1\text{-loop}} = \frac{m_B}{m_b} \frac{(s_B^4 - s_{qB}^4)}{48\pi^2} \left[-\frac{1}{\epsilon} + \gamma_E - 1 - \log(4\pi) + \log\left(\frac{m_q^2}{\mu^2}\right) \right]. \quad (2.35)$$

The UV divergence appearing in the matrix element Eq. (2.35) is removed by the standard renormalisation procedure in the $\overline{\text{MS}}$ scheme. The corresponding divergence is absorbed into the renormalisation constant of the dimension-seven four-quark operator, from which the associated anomalous dimension is obtained. The renormalised matrix element is then evaluated at the matching scale $\mu = m_b$. For further details on the renormalisation procedure and the derivation of the anomalous dimensions, we refer the reader to Ref. [62]. The same procedure applies to all remaining dimension-seven four-quark operators, for both the single- and double-gluon topologies. The resulting renormalised matrix elements read

$$\langle R_0^q \rangle^{\text{ren}} = 0, \quad \langle R_1^q \rangle^{\text{ren}} = \frac{m_B}{24\pi^2 m_b} \left[\log\left(\frac{m_b^2}{m_q^2}\right) - 1 \right] (2r_E^{(1)} - 3r_G^{(1)}), \quad (2.36)$$

$$\langle R_2^q \rangle^{\text{ren}} = \frac{m_B}{24\pi^2 m_b} \log\left(\frac{m_b^2}{m_q^2}\right) (2r_E^{(1)} - 3r_G^{(1)}), \quad (2.37)$$

$$\langle R_3^q \rangle^{\text{ren}} = \frac{m_B}{12\pi^2 m_b} (2r_E^{(1)} - r_G^{(1)}), \quad (2.38)$$

and for the colour-rearranged operators we obtain

$$\begin{aligned} \langle \tilde{R}_0^q \rangle^{\text{ren}} &= \frac{m_B}{96\pi^2 m_b} \left[-2 \log\left(\frac{m_b^2}{m_q^2}\right) (4s_B^4 - 4s_E^4 + s_{qB}^4) \right. \\ &\quad \left. + 8r_E^4 - 2r_G^4 - 4s_B^4 + 4s_E^4 - s_{qB}^4 \right], \end{aligned} \quad (2.39)$$

$$\begin{aligned} \langle \tilde{R}_1^q \rangle^{\text{ren}} &= \frac{m_B}{48\pi^2 m_b} \left[\log\left(\frac{m_b^2}{m_q^2}\right) (8r_E^4 - 2r_G^4 + 4s_B^4 - 4s_E^4 + s_{qB}^4) \right. \\ &\quad \left. - 8r_E^4 + 2r_G^4 + 4s_B^4 - 4s_E^4 + s_{qB}^4 \right], \end{aligned} \quad (2.40)$$

$$\langle \tilde{R}_2^q \rangle^{\text{ren}} = \frac{m_B}{48\pi^2 m_b} \log\left(\frac{m_b^2}{m_q^2}\right) [8r_E^4 - 2r_G^4 - 4s_E^4 - s_{qB}^4], \quad (2.41)$$

$$\langle \tilde{R}_3^q \rangle^{\text{ren}} = \frac{m_B}{12\pi^2 m_b} r_G^4. \quad (2.42)$$

Note that the operators R_i receive contributions only from the colour-singlet two-quark operators defined in Eq. (B.4), while the colour-rearranged operators $\tilde{R}_i$ contain contributions from the combined colour-singlet and colour-octet structures given in Eq. (B.3).

The cancellation of the IR logarithms is implemented at the level of the physical dimension-seven contribution to the width. The four-quark contributions at order $1/m_b$ are first written in terms of the operators R_i^q and $\tilde{R}_i^q$, following Ref. [80]. In a second step, the renormalised gluonic matrix elements of these operators, derived above, are inserted into the corresponding expressions for $\delta_{1/m}^q$. This generates additional contributions to the same dimension-seven two-quark HQE matrix elements that appear in the direct two-quark matching.

The finite coefficients quoted in the next section are obtained by adding these induced four-quark contributions to the direct two-quark contributions from $\mathcal{T}_{11}^{(q_1\bar{q}_2q_3)}$, $\mathcal{T}_{12}^{(q_1\bar{q}_2q_3)}$, and $\mathcal{T}_{22}^{(q_1\bar{q}_2q_3)}$. In this sum, the logarithms depending on the auxiliary light-quark masses cancel, leaving regulator-independent matching coefficients for the dimension-seven two-quark operator basis.

In addition to the renormalised matrix elements of the dimension-seven four-quark operators discussed above, the expansion of the dimension-six Darwin contribution, parameterized by the HQE matrix element ρ_D^3 , to $\mathcal{O}(1/m_b^4)$ also induces mixing into the dimension-seven HQE basis. Consequently, the renormalised matrix elements of the corresponding dimension-six four-quark operators are likewise required. Following Ref. [81], we employ the operator basis

$$\mathcal{O}_1^q = 4(\bar{b}\gamma_\mu P_L q)(\bar{q}\gamma^\mu P_L b), \quad \mathcal{O}_2^q = 4(\bar{b}P_L q)(\bar{q}P_R b), \quad (2.43)$$

whose renormalised matrix elements, expanded through the dimension-seven HQE basis, are given by

$$\begin{aligned} \langle \mathcal{O}_1^q \rangle^{\text{ren}} &= \frac{m_B}{3\pi^2} \left[\log\left(\frac{m_b^2}{m_q^2}\right) - 1 \right] \rho_D^3 \\ &\quad + \frac{m_B}{24\pi^2 m_b} \left[\log\left(\frac{m_b^2}{m_q^2}\right) - 1 \right] (4r_G^4 + 4s_B^4 - s_{qB}^4) + \mathcal{O}\left(\frac{1}{m_b^2}\right), \end{aligned} \quad (2.44)$$

$$\langle \mathcal{O}_2^q \rangle^{\text{ren}} = -\frac{m_B}{6\pi^2} \log\left(\frac{m_b^2}{m_q^2}\right) \rho_D^3 - \frac{m_B}{48\pi^2 m_b} \log\left(\frac{m_b^2}{m_q^2}\right) (4r_G^4 - 4s_B^4 + s_{qB}^4) + \mathcal{O}\left(\frac{1}{m_b^2}\right). \quad (2.45)$$

As for the dimension-seven four-quark operators, the renormalised matrix elements given above are first inserted into the corresponding expressions of Ref. [81]. The contributions generated by the expansion of the dimension-six Darwin term are then added to these expressions. Finally, combining the resulting four-quark contributions with the direct two-quark matching leads to the cancellation of all logarithms depending on the auxiliary light-quark masses, yielding the finite analytical results presented in the following section.

2.2 Analytical Results

The dimension-seven two-quark contributions to the inclusive non-leptonic decay width $b \rightarrow q_1\bar{q}_2q_3$ can be written as

$$\Gamma_{1/m_b^4}^{(q_1\bar{q}_2q_3)}(B) = \tilde{\Gamma}_0 \frac{1}{m_b^4} \left[N_c C_1^2 \mathcal{W}_{11}^{(q_1\bar{q}_2q_3)}(\rho) + C_1 C_2 \mathcal{W}_{12}^{(q_1\bar{q}_2q_3)}(\rho) + N_c C_2^2 \mathcal{W}_{22}^{(q_1\bar{q}_2q_3)}(\rho) \right], \quad (2.46)$$

where

$$\tilde{\Gamma}_0 = \frac{G_F^2 m_b^5 |V_{q_1 b}|^2 |V_{q_2 q_3}|^2}{192\pi^3}, \quad (2.47)$$

$N_c = 3$ denotes the number of colours and $\rho = m_c^2/m_b^2$. The functions $\mathcal{W}_{ij}^{(q_1 \bar{q}_2 q_3)}(\rho)$ encode the contributions associated with the operator products $Q_i \otimes Q_j$. Each of them can be expressed as a linear combination of the dimension-seven HQE matrix elements introduced in Eqs. (B.3) and (B.4). The explicit expressions are given below.

2.2.1 Results for the $b \rightarrow c\bar{u}d$ Transition

$$\begin{aligned} \mathcal{W}_{11}^{(c\bar{u}d)} = & -\frac{8}{9}r_E^4(9\rho^4 - 20\rho^3 + 9\rho^2 + 6\log\rho + 2) + \frac{4}{9}r_G^4(3\rho^4 - 7\rho^3 + 9\rho^2 - 21\rho + 12\log\rho + 16) \\ & - \frac{2}{9}s_E^4(9\rho^4 - 20\rho^3 + 36\rho - 12\log\rho - 25) + \frac{2}{3}s_B^4(\rho - 1)^3(5\rho + 1) \\ & - \frac{1}{36}s_{qB}^4(3\rho^4 - 16\rho^3 + 36\rho^2 - 48\rho + 12\log\rho + 25) \\ & + \frac{2}{9N_c}r_E^{(1)}(-26\rho^3 - 24\rho^3\log\rho + 48(\rho^3 - 1)\log(1 - \rho) + 57\rho^2 - 12\rho - 19) \\ & - \frac{1}{N_c}r_G^{(1)}(-8\rho^3\log\rho + 16(\rho^3 - 1)\log(1 - \rho) - (\rho - 1)^2(6\rho + 1)). \end{aligned} \quad (2.48)$$

$$\begin{aligned} \mathcal{W}_{22}^{(c\bar{u}d)} = & -\frac{8}{9}r_E^4(9\rho^4 - 62\rho^3 + 66\rho^2 - 6(4\rho - 3)\rho^2\log\rho + 12(4\rho^3 - 3\rho^2 - 1)\log(1 - \rho) - 12\rho - 1) \\ & + \frac{2}{9}r_G^4(6\rho^4 - 23\rho^3 - 6\rho^3\log\rho + 12(\rho^3 + 3\rho - 4)\log(1 - \rho) + 3\rho^2 - 9\rho + 23) \\ & - \frac{2}{9}s_E^4(9\rho^4 - 68\rho^3 + 102\rho^2 + 12(3 - 2\rho)\rho^2\log\rho - 12\rho \\ & \quad + 24(\rho - 1)^2(2\rho + 1)\log(1 - \rho) - 31) \\ & + \frac{2}{3}s_B^4(5\rho^4 - 23\rho^3 + 30\rho^2 - 6(\rho - 2)\rho^2\log\rho - 5\rho + 12(\rho - 1)^2\rho\log(1 - \rho) - 7) \\ & - \frac{1}{36}s_{qB}^4(3\rho^4 - 94\rho^3 + 96\rho^2 - 12(\rho - 3)\rho^2\log\rho - 18\rho + 24(\rho - 1)^3\log(1 - \rho) + 13) \\ & + \frac{4}{9N_c}r_E^{(1)}(18(\rho^2 - 1)\log(1 - \rho) + (3 - 9\rho^2)\log\rho + 2(7\rho^3 - 18\rho^2 + 18\rho - 7)) \\ & - \frac{2}{3N_c}r_G^{(1)}(18(\rho^2 - 1)\log(1 - \rho) + (3 - 9\rho^2)\log\rho + (5\rho + 1)(\rho - 1)^2). \end{aligned} \quad (2.49)$$

$$\begin{aligned} \mathcal{W}_{12}^{(c\bar{u}d)} = & -\frac{16}{9}r_E^4(9\rho^4 - 76\rho^3 + 84\rho^2 + (-24\rho^3 + 36\rho^2 + 6)\log\rho \\ & \quad - 12\rho + 24(\rho - 1)^2(2\rho + 1)\log(1 - \rho) - 5) \\ & + \frac{8}{9}r_G^4(24(\rho^3 - 3\rho + 2)\log(1 - \rho) + (3 - 12\rho^3)\log\rho + (3\rho^2 - 20\rho - 10)(\rho - 1)^2) \\ & - \frac{4}{9}s_E^4(48(\rho^3 - 1)\log(1 - \rho) - 12(2\rho^3 + 1)\log\rho + (9\rho + 11)(\rho - 1)^3) \\ & + \frac{4}{9}s_B^4(15\rho^4 - 92\rho^3 + 6\rho^2 + 12(2\rho^3 + 9\rho^2 - 12\rho + 1)\log(1 - \rho) \\ & \quad - 6(2\rho^3 + 9\rho^2 + 1)\log\rho + 96\rho - 25) \\ & - \frac{1}{6}s_{qB}^4(\rho - 1)(\rho^3 + \rho^2 + 31\rho - 96\log(1 - \rho) + 87). \end{aligned} \quad (2.50)$$

2.2.2 Results for the $b \rightarrow c\bar{c}s$ Transition

For the $b \rightarrow c\bar{c}s$ channel, we define $\xi_1 = \sqrt{1-4\rho}$ and $\xi_2 = \log\left(\frac{1-\xi_1}{1+\xi_1}\right)$, which will be used throughout the following expressions.

$$\begin{aligned}
\mathcal{W}_{11}^{(c\bar{c}s)} = & \frac{16}{9} \frac{r_E^4}{\xi_1^3} \left[\rho(80\rho^2 - 19\rho - 7)\xi_1\xi_2 - (54\rho^3 + 39\rho^2 - 4\rho + 1)\xi_1^4 \right. \\
& \left. - (432\rho^5 + 132\rho^4 - 52\rho^3 - \rho^2 - 19\rho + 3)\xi_1\xi_2 \right] \\
& + \frac{2r_G^4}{9\xi_1} \left[-288\rho^4 - 144\rho^3 + 74\rho^2 - 133\rho + 32 - 144\xi_1\rho^4\xi_2 + 180\xi_1\rho^3\log(1+\xi_1) \right. \\
& + 96\xi_1\rho^3\xi_2 - 18\xi_1\rho^2\log(1+\xi_1) + 14\xi_1\rho^2\xi_2 - 9\xi_1(20\rho^3 - 2\rho^2 + 2\rho + 1)\log(1-\xi_1) \\
& \left. - 8\xi_1(4\rho + 5)\rho\xi_2 + 18\xi_1\rho\log(1+\xi_1) + 40\xi_1\rho\xi_2 + 9\xi_1\log(1+\xi_1) + 33\xi_1\xi_2 \right] \\
& - \frac{2s_E^4}{9\xi_1} \left[-432\rho^4 - 348\rho^3 + 218\rho^2 + 74\rho - 25 \right. \\
& \left. - 4\xi_1(54\rho^4 + 48\rho^3 - 2\rho^2 + 2\rho + 3)\xi_2 + 8\xi_1(8\rho + 1)\rho\xi_2 \right] \\
& + \frac{2s_B^4}{9\xi_1} \left[-720\rho^4 - 156\rho^3 + 264\rho^2 - 33\rho - 3 \right. \\
& - 360\xi_1\rho^4\xi_2 - 60\xi_1\rho^3\log(1+\xi_1) - 168\xi_1\rho^3\xi_2 \\
& + 18\xi_1\rho^2\log(1+\xi_1) + 34\xi_1\rho^2\xi_2 + 3\xi_1(20\rho^3 - 6\rho^2 - 6\rho - 1)\log(1-\xi_1) \\
& \left. + 4\xi_1(14\rho - 5)\rho\xi_2 + 18\xi_1\rho\log(1+\xi_1) + 20\xi_1\rho\xi_2 + 3\xi_1\log(1+\xi_1) + 3\xi_1\xi_2 \right] \\
& - \frac{s_B^4}{36} \left[-72\rho^4\xi_2 + 36\xi_1\rho^3 - 120\rho^3\log(1+\xi_1) - 144\rho^3\xi_2 + 18\xi_1\rho^2 + 36\rho^2\log(1+\xi_1) \right. \\
& + 60\rho^2\xi_2 + 6(20\rho^3 - 6\rho^2 - 6\rho - 1)\log(1-\xi_1) - 52\xi_1\rho + 25\xi_1 \\
& \left. + 48(\rho - 1)\rho\xi_2 + 36\rho\log(1+\xi_1) + 48\rho\xi_2 + 6\log(1+\xi_1) + 18\xi_2 \right] \\
& + \frac{2r_E^{(1)}}{9N_c\xi_1^5} \left[-\xi_1^4 \left(48\rho^3 - 44\rho^2 + 40\rho + 1 + 24(4\rho^2 - 2\rho + 1)\log\xi_1 \right. \right. \\
& \left. \left. - 12(4\rho^2 - 2\rho + 1)\log\rho \right) \right. \\
& + 2\xi_1(200\rho^4 - 12\rho^3 + 90\rho^2 - 44\rho + 9)\xi_2 \\
& \left. - 4\xi_1(96\rho^5 + 340\rho^4 - 144\rho^3 + 39\rho^2 - 10\rho + 3)\xi_2 \right] \\
& - \frac{r_G^{(1)}}{3N_c\xi_1^5} \left[-3\xi_1^4 \left(16\rho^3 + 4\rho^2 + 28\rho - 5 + 8(4\rho^2 - 2\rho + 1)\log\xi_1 \right. \right. \\
& \left. \left. - 4(4\rho^2 - 2\rho + 1)\log\rho \right) \right. \\
& + 2\xi_1(176\rho^4 - 24\rho^3 + 138\rho^2 - 56\rho + 9)\xi_2 \\
& \left. - 4\xi_1(96\rho^5 + 328\rho^4 - 150\rho^3 + 63\rho^2 - 16\rho + 3)\xi_2 \right]. \tag{2.51}
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_{22}^{(c\bar{c}s)} = & -\frac{8r_E^4}{9\xi_1^5} \left[2\rho(356\rho^3 - 66\rho^2 + 27\rho + 7)\xi_1\xi_2 - \xi_1^4(432\rho^4 + 156\rho^3 - 98\rho^2 + 22\rho + 1 \right. \\
& + 24(4\rho^2 - 2\rho + 1)\log\xi_1 - 12(4\rho^2 - 2\rho + 1)\log\rho) \\
& \left. - 2(1728\rho^6 - 96\rho^5 + 368\rho^4 - 162\rho^3 + 141\rho^2 - 41\rho + 6)\xi_1\xi_2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{2r_G^4}{9\xi_1^5} \left[8\rho(-16\rho^3 + 6\rho^2 + 9\rho + 1)\xi_1\xi_2 + \xi_1^4(-288\rho^4 - 24\rho^3 + 116\rho^2 - 112\rho + 29 \right. \\
& \quad + 12\xi_1\xi_3\rho - 12\xi_1\xi_3 - 48(2\rho^2 - 7\rho + 2)\log\xi_1 + 24(2\rho^2 - 7\rho + 2)\log\rho) \\
& \quad \left. - 8(288\rho^6 - 96\rho^5 - 22\rho^4 - 135\rho^3 + 177\rho^2 - 56\rho + 6)\xi_1\xi_2 \right] \\
& - \frac{2s_E^4}{9\xi_1} \left[-432\rho^4 - 156\rho^3 + 338\rho^2 + 38\rho - 31 + 24(8\rho^2 + 2\rho - 1)\log\left(\frac{\rho}{1-4\rho}\right) \right. \\
& \quad \left. - 8\xi_1(27\rho^4 + 12\rho^3 + 17\rho^2 + \rho - 3)\xi_2 + 8\xi_1(8\rho + 1)\rho\xi_2 \right] \\
& + \frac{2s_B^4}{3\xi_1} \left[-240\rho^4 - 28\rho^3 + 110\rho^2 - 16\rho - 3 + 12\xi_1\rho^2\xi_2 + 24\rho(1-4\rho)\log\left(\frac{\rho}{1-4\rho}\right) \right. \\
& \quad \left. - 12\xi_1(10\rho^3 + 2\rho^2 + 3\rho - 2)\rho\xi_2 + 4\xi_1\xi_3(\rho - 1) \right] \\
& - \frac{s_B^4}{36\xi_1} \left[-144\rho^4 - 84\rho^3 + 310\rho^2 - 182\rho + 37 + 24(4\rho - 1)^2\log\left(\frac{\rho}{1-4\rho}\right) + 24\xi_1\xi_3(\rho - 1) \right. \\
& \quad \left. - 8\xi_1(9\rho^4 + 6\rho^3 + 10\rho^2 - 19\rho + 3)\xi_2 + 8\xi_1(\rho - 1)\rho\xi_2 \right] \\
& - \frac{4r_E^{(1)}}{9N_c\xi_1^3} \left[-(228\rho^2 - 16\rho - 5)\xi_1^4 + (608\rho^3 - 76\rho^2 - 10\rho - 9)\xi_1\xi_2 \right. \\
& \quad \left. - (1824\rho^4 - 280\rho^3 + 32\rho^2 - 58\rho + 3)\xi_1\xi_2 \right] \\
& + \frac{2r_G^{(1)}}{3N_c\xi_1^3} \left[-2(6\rho^2 + \rho + 5)\xi_1^4 + (32\rho^3 - 16\rho^2 + 20\rho - 9)\xi_1\xi_2 \right. \\
& \quad \left. - (96\rho^4 + 8\rho^3 - 16\rho^2 - 28\rho + 3)\xi_1\xi_2 \right]. \tag{2.52}
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_{12}^{(c\bar{c}s)} = & -\frac{32r_E^4}{9\xi_1^5} \left[-\xi_1^4(216\rho^4 + 486\rho^3 - 155\rho^2 - 14\rho + 7 + 12(4\rho^2 - 2\rho + 1)\log\xi_1 \right. \\
& - 6(4\rho^2 - 2\rho + 1)\log\rho) \\
& + (1444\rho^4 - 366\rho^3 - 90\rho^2 + 74\rho - 9)\xi_1\xi_2 \\
& \left. - (1728\rho^6 + 3168\rho^5 - 752\rho^4 + 30\rho^3 + 84\rho^2 - 22\rho + 3)\xi_1\xi_2 \right] \\
& + \frac{8r_G^4}{9\xi_1^5} \left[-2304\rho^6 - 5568\rho^5 + 5648\rho^4 - 164\rho^3 - 936\rho^2 + 268\rho - 22 \right. \\
& + 6(64\rho^4 - 64\rho^3 + 36\rho^2 - 10\rho + 1)\log\left(\frac{\rho}{1-4\rho}\right) \\
& + \xi_1(-576\rho^5 + 864\rho^4 - 612\rho^3 + 468\rho^2 - 162\rho + 18)\log(1 + \xi_1) \\
& + 18\xi_1^5(2\rho^3 - 2\rho^2 + \rho - 1)\log(1 - \xi_1) \\
& \left. + \xi_1(-1152\rho^6 - 3456\rho^5 + 584\rho^4 - 516\rho^3 + 318\rho^2 - 20\rho - 6)\xi_2 \right] \\
& + \frac{4s_E^4}{9\xi_1^3} \left[-1728\rho^5 + 1728\rho^4 + 868\rho^3 - 162\rho^2 - 174\rho + 35 + 24(32\rho^3 - 6\rho + 1)\log\left(\frac{\rho}{1-4\rho}\right) \right. \\
& + 144\xi_1\rho^3(4\rho - 1)\log(1 + \xi_1) + 144\xi_1^3\rho^3\log(1 - \xi_1) \\
& \left. + \xi_1(-864\rho^5 + 1368\rho^4 - 576\rho^3 + 72\rho^2 + 192\rho - 48)\xi_2 \right] \\
& + \frac{4s_B^4}{9\xi_1} \left[-720\rho^4 - 84\rho^3 + 26\rho^2 + 164\rho - 43 + 12(8\rho^2 + 2\rho - 1)\log\left(\frac{\rho}{1-4\rho}\right) \right. \\
& \left. + \xi_1(-360\rho^4 - 72\rho^3 - 144\rho^2 + 36\rho + 6)\xi_2 \right] \\
& - \frac{s_B^4}{6\xi_1} \left[-48\rho^4 + 292\rho^3 + 18\rho^2 + 258\rho - 67 + \xi_1(-24\rho^4 + 144\rho^3 + 24\rho - 48)\xi_2 \right]. \tag{2.53}
\end{aligned}$$

2.3 Closing Remarks

In this chapter, we have completed the leading-order calculation of the dimension-seven two-quark contribution $\Gamma_7^{(0)}$ to inclusive non-leptonic b -hadron decays for the CKM-favoured channels $b \rightarrow \bar{c}ud$ and $b \rightarrow \bar{c}\bar{s}s$. The resulting analytical matching coefficients provide the complete $\mathcal{O}(1/m_b^4)$ two-quark corrections for these channels and extend the non-leptonic HQE by one order beyond the dimension-six Darwin contribution.

A distinctive feature of the non-leptonic calculation at leading order in QCD is the separate appearance of the colour-singlet matrix elements $r_G^{(1)}$ and $r_E^{(1)}$ defined in Eq. (B.4). In the semileptonic case, the colour-singlet and colour-octet components enter only through specific combinations, whereas the non-leptonic contributions probe $r_G^{(1)}$ and $r_E^{(1)}$ individually. These matrix elements contribute only to the terms proportional to C_1^2 and C_2^2 and are absent from the interference term proportional to $C_1 C_2$ at $\mathcal{O}(1/m_b^4)$.

As an important consistency check, the corresponding semileptonic result (see, e.g., Ref. [79]) is recovered directly from the $b \rightarrow \bar{c}ud$ expressions by taking $N_c \rightarrow 1$, $C_1 \rightarrow 1$, and $C_2 \rightarrow 0$, setting $r_G^{(1)} = r_E^{(1)} = 0$, and replacing the CKM factors accordingly. In particular, the semileptonic contribution follows from $\mathcal{W}_{11}^{(\bar{c}ud)}$ after omitting the terms proportional to $r_G^{(1)}$ and $r_E^{(1)}$.

The results derived here provide the perturbative input required to incorporate dimension-seven two-quark effects systematically into precision predictions for B -meson decay widths and lifetime ratios. Together with improved determinations of the corresponding non-perturbative matrix elements, they will allow the numerical impact of the $\mathcal{O}(1/m_b^4)$ corrections to be assessed and provide a direct test of the convergence of the HQE for inclusive non-leptonic decays at this order.

Chapter 3

Model-Independent BSM Contributions up to the Darwin Term

*“These BSM calculations are getting rather involved.
After all, the Standard Model is not so bad, right?”*

— Thomas Mannel

The previous chapter focused on the SM description of inclusive non-leptonic b -hadron decays within the HQE. We now turn to the complementary question of how the same framework can be used to search for BSM physics. In particular, new heavy degrees of freedom—too massive to be produced directly—can leave imprints at low energies by modifying the short-distance structure of the effective weak Hamiltonian. This framework also provides a powerful and model-independent tool for BSM searches: new heavy particles manifest as either modified Wilson coefficients C_i of existing SM operators or as entirely new local operators O_i^{BSM} that respect the symmetries of the low-energy theory. These additional operators can contribute to observables such as the total inclusive width or lifetime ratios, allowing one to probe new-physics effects indirectly.

In this chapter, we extend the HQE to include the most general dimension-six $|\Delta B| = 1$ four-fermion operators contributing to non-leptonic b -hadron decays. We determine their leading-order contributions to the inclusive decay width, comprising the leading free-quark contribution Γ_3 , the dimension-five chromomagnetic correction Γ_5 , and the dimension-six two-quark Darwin term Γ_6 . This study is motivated by the sizable numerical contribution of the Darwin term in the SM, suggesting that the corresponding contributions arising from a generalised effective Hamiltonian may likewise have an important phenomenological impact. We further derive the previously uncalculated weak-annihilation (WA) topology of the dimension-six $\Delta B = 0$ four-quark contribution $\tilde{\Gamma}_6$. As a direct consequence of this general analysis, we also obtain new results for the contributions of the SM QCD-penguin operators at dimension five and dimension six, which follow from the general BSM operator basis after appropriate Fierz transformations. The resulting expressions provide a model-independent parametrisation of short-distance new-physics effects that can be readily applied to a broad class of ultraviolet completions and used to constrain their Wilson coefficients through precision measurements of inclusive b -hadron decay widths and lifetime ratios.

3.1 Generalised Effective Hamiltonian

To parameterise possible BSM contributions in a model-independent manner, we consider the most general effective Hamiltonian describing non-leptonic $|\Delta B| = 1$ transitions $b \rightarrow q_1 \bar{q}_2 q_3$, with $q_1, q_2 = u, c$ and $q_3 = d, s$. It is obtained by supplementing the SM operator basis with all additional dimension-six four-fermion operators [82, 83]:

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{q_1 b} V_{q_2 q_3}^* \sum_{i=1}^{10} \left[\mathcal{C}_i(\mu_b) \mathcal{Q}_i^{q_1 q_2 q_3} + \mathcal{C}'_i(\mu_b) \mathcal{Q}'_i^{q_1 q_2 q_3} \right] + \text{h.c.} \quad (3.1)$$

Here, the four-quark operators $\mathcal{Q}_i^{q_1 q_2 q_3}$ are defined as

$$\mathcal{Q}_1^{q_1 q_2 q_3} = (\bar{q}_1^i \gamma_\mu P_L b^i) (\bar{q}_3^j \gamma^\mu P_L q_2^j), \quad \mathcal{Q}_2^{q_1 q_2 q_3} = (\bar{q}_1^i \gamma_\mu P_L b^j) (\bar{q}_3^j \gamma^\mu P_L q_2^i), \quad (3.2)$$

$$\mathcal{Q}_3^{q_1 q_2 q_3} = (\bar{q}_1^i \gamma_\mu P_R b^i) (\bar{q}_3^j \gamma^\mu P_L q_2^j), \quad \mathcal{Q}_4^{q_1 q_2 q_3} = (\bar{q}_1^i \gamma_\mu P_R b^j) (\bar{q}_3^j \gamma^\mu P_L q_2^i), \quad (3.3)$$

$$\mathcal{Q}_5^{q_1 q_2 q_3} = (\bar{q}_1^i P_L b^i) (\bar{q}_3^j P_R q_2^j), \quad \mathcal{Q}_6^{q_1 q_2 q_3} = (\bar{q}_1^i P_L b^j) (\bar{q}_3^j P_R q_2^i), \quad (3.4)$$

$$\mathcal{Q}_7^{q_1 q_2 q_3} = (\bar{q}_1^i P_R b^i) (\bar{q}_3^j P_R q_2^j), \quad \mathcal{Q}_8^{q_1 q_2 q_3} = (\bar{q}_1^i P_R b^j) (\bar{q}_3^j P_R q_2^i), \quad (3.5)$$

$$\mathcal{Q}_9^{q_1 q_2 q_3} = (\bar{q}_1^i \sigma_{\mu\nu} P_R b^i) (\bar{q}_3^j \sigma^{\mu\nu} P_R q_2^j), \quad \mathcal{Q}_{10}^{q_1 q_2 q_3} = (\bar{q}_1^i \sigma_{\mu\nu} P_R b^j) (\bar{q}_3^j \sigma^{\mu\nu} P_R q_2^i), \quad (3.6)$$

where i, j denote colour indices and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$. The corresponding primed operators, $\mathcal{Q}'_i^{q_1 q_2 q_3}$, are obtained by interchanging the left- and right-handed chiral projectors,

$$\mathcal{Q}'_i^{q_1 q_2 q_3} = \mathcal{Q}_i^{q_1 q_2 q_3} \Big|_{P_L \leftrightarrow P_R}. \quad (3.7)$$

The Wilson coefficients $\mathcal{C}_i^{(\prime)}$ in Eq. (3.1) encode the short-distance dynamics and are, in general, complex-valued. Their dependence on the flavour combination $(q_1 q_2 q_3)$ is understood and will be suppressed throughout this chapter whenever no ambiguity arises.

The generalised effective Hamiltonian in Eq. (3.1) contains the SM Hamiltonian in Eq. (1.9) as a special case. In particular, the operators $\mathcal{Q}_1^{q_1 q_2 q_3}$ and $\mathcal{Q}_2^{q_1 q_2 q_3}$ coincide with the SM current-current operators $Q_1^{q_1 q_2 q_3}$ and $Q_2^{q_1 q_2 q_3}$,

$$Q_1^{q_1 q_2 q_3} = \mathcal{Q}_1^{q_1 q_2 q_3}, \quad Q_2^{q_1 q_2 q_3} = \mathcal{Q}_2^{q_1 q_2 q_3}. \quad (3.8)$$

The SM QCD-penguin operators Q_{3-6}^{qq3} in Eq. (1.9) with $q = u, c$ can likewise be mapped onto the generalised operator basis. Using Fierz transformations, one finds

$$Q_3^{qq3} = \mathcal{Q}_2^{qq3}, \quad Q_4^{qq3} = \mathcal{Q}_1^{qq3}, \quad (3.9)$$

$$Q_5^{qq3} = -2 \mathcal{Q}_6^{qq3}, \quad Q_6^{qq3} = -2 \mathcal{Q}_5^{qq3}. \quad (3.10)$$

The remaining penguin contributions with $q = d, s$ are obtained from the $q = u$ coefficient functions upon neglecting the light-quark masses, as discussed in Section C.4.

Correspondingly, the Wilson coefficients $\mathcal{C}_i^{(\prime)}$ can be decomposed into their SM and BSM contributions,

$$\mathcal{C}_i^{(\prime)} = \mathcal{C}_{i,\text{SM}}^{(\prime)} + \mathcal{C}_{i,\text{BSM}}^{(\prime)}, \quad (3.11)$$

where the SM part is fixed by the underlying theory, while the BSM contribution is, in general, a complex quantity encoding the effects of new short-distance physics. For example, $\mathcal{C}_{1,\text{SM}} = C_1$ and $\mathcal{C}'_{1,\text{SM}} = 0$. With this decomposition, the double insertion of the generalised effective Hamiltonian in Eq. (3.1) into Eq. (1.13) naturally generates all contributions to the inclusive decay width: the purely SM terms, the purely BSM terms, and the SM–BSM interference terms. We therefore adopt the generalised effective Hamiltonian in Eq. (3.1) as the starting point of our analysis, and all results presented in this chapter are expressed in terms of this operator basis.

Despite the considerable progress achieved in recent years, a complete HQE analysis based on the generalised effective Hamiltonian in Eq. (3.1) has not yet been available. Existing studies have focused primarily on the leading-power contributions and on the dimension-six spectator effects. In particular, the matching coefficients of the four-quark operators, $\tilde{\Gamma}_6^{(0)}$, have been determined for the $b \rightarrow c\bar{c}s$ transition in Refs. [81, 84, 85] and for the $b \rightarrow c\bar{u}d$ transition in Ref. [81]. The leading free-quark contribution to the generalised operator basis was subsequently derived for the $b \rightarrow c\bar{u}d$ channel at leading order in QCD in Ref. [86], while the corresponding hadronic matrix elements of the dimension-six four-quark operators were obtained in the HQET limit using three-loop sum rules in Ref. [38].

The results presented in this chapter complete the HQE matching for the generalised effective Hamiltonian through $\mathcal{O}(1/m_b^3)$, thereby providing the missing ingredients required for a complete model-independent description of inclusive non-leptonic b -hadron decays.

3.2 Calculation Procedure

The matching calculation follows the same methodology presented in detail in Chapter 2, and we only summarize the main steps here. We start from the time-ordered product in Eq. (1.13) and evaluate the forward-scattering amplitude by expanding the internal quark propagators in the external gluon field using the FS gauge.¹ Since the present analysis is restricted to operators up to dimension six, the background-field expansion is truncated accordingly. After evaluating the resulting master loop integrals (see Section B.2), the amplitudes are expanded in the external momenta and matched onto the HQE operator basis defined in Section B.1. The matching onto the local HQE operators is performed using the trace formalism of Refs. [79, 87], yielding the Wilson coefficients of the dimension-three, dimension-five, dimension-six two-quark, and dimension-six four-quark operators.

As discussed in Chapter 2, the master loop integrals entering the matching calculation are UV finite and can therefore be evaluated directly in four dimensions. Starting at the dimension-six two-quark level, however, IR singularities arise from the massless light-quark propagators. We regulate these singularities by retaining finite light-quark masses throughout the calculation, which gives rise to logarithmic

¹In contrast to the SM calculation in Chapter 2, where only terms containing an odd number of Dirac matrices in the propagator expansion contribute, the general BSM operator basis involves different Dirac structures for which both odd and even contributions are relevant. We therefore employ the full background-field expansion of the quark propagator, retaining terms with up to one covariant derivative acting on the gluon field-strength tensor, as given, e.g., in Refs. [62, 63].

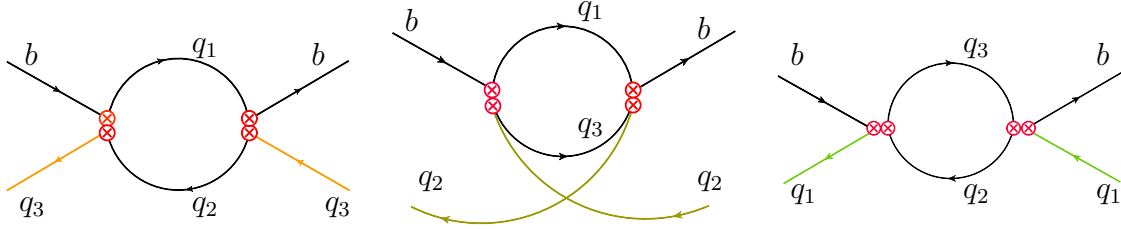


Figure 3.1: Diagrammatic representation of the $\Delta B = 0$ four-quark operator contributions in the HQE for the transition $b \rightarrow q_1 \bar{q}_2 q_3$: weak exchange (WE), Pauli interference (PI), and weak annihilation (WA).

terms of the form $\log(m_q^2/m_b^2)$ in the matching coefficients. The appearance of these IR logarithms reflects the mixing of the dimension-six two-quark operators into the $\Delta B = 0$ dimension-six four-quark operators under renormalisation. To obtain IR-finite Wilson coefficients, we explicitly compute the corresponding one-loop renormalisation of the four-quark operators and combine it with the bare matching coefficients. The logarithmic dependence on the light-quark masses then cancels exactly, yielding finite matching coefficients for the dimension-six HQE operators.

The $\Delta B = 0$ four-quark operators required for the renormalisation of the dimension-six two-quark matching coefficients were determined in Ref. [81]. Employing the same generalised effective Hamiltonian in Eq. (3.1), they obtained the operator basis

$$\mathcal{O}_1 = 4 (\bar{b} \gamma_\mu P_L q) (\bar{q} \gamma^\mu P_L b), \quad \mathcal{O}_2 = 4 (\bar{b} P_L q) (\bar{q} P_R b), \quad (3.12)$$

$$\mathcal{O}_3 = 4 (\bar{b} \gamma_\mu P_L t^a q) (\bar{q} \gamma^\mu P_L t^a b), \quad \mathcal{O}_4 = 4 (\bar{b} P_L t^a q) (\bar{q} P_R t^a b), \quad (3.13)$$

$$\mathcal{O}_5 = 4 (\bar{b} \gamma_\mu P_L q) (\bar{q} \gamma^\mu P_R b), \quad \mathcal{O}_6 = 4 (\bar{b} P_L q) (\bar{q} P_L b), \quad (3.14)$$

$$\mathcal{O}_7 = 4 (\bar{b} \gamma_\mu P_L t^a q) (\bar{q} \gamma^\mu P_R t^a b), \quad \mathcal{O}_8 = 4 (\bar{b} P_L t^a q) (\bar{q} P_L t^a b), \quad (3.15)$$

Together with the corresponding chirality-flipped operators $\mathcal{O}'_i$, obtained from $\mathcal{O}_i$ by the replacement $P_L \leftrightarrow P_R$, these operators constitute the complete $\Delta B = 0$ operator basis.²

The generalised effective Hamiltonian in Eq. (3.1) contributes to the four inclusive non-leptonic transitions $b \rightarrow c \bar{u} d$, $b \rightarrow c \bar{c} s$, $b \rightarrow u \bar{c} s$, and $b \rightarrow u \bar{u} d$. Although the matching procedure is identical for all channels, the cancellation of IR singularities depends on the underlying spectator topology.

For the $b \rightarrow c \bar{u} d$ and $b \rightarrow c \bar{c} s$ transitions, the IR logarithms are completely cancelled by including the one-loop renormalisation of the $\Delta B = 0$ four-quark operators arising from the Pauli interference (PI) and weak exchange (WE) topologies shown in the left and middle panels of Figure 3.1. These contributions were calculated

²In principle, tensor four-quark operators may also be included. At dimension six, however, they are not independent: using the b -quark equations of motion, they can be reduced to the remaining Dirac structures up to $\mathcal{O}(1/m_b)$ corrections [81]. Moreover, their renormalised soft-gluon corrections vanish.

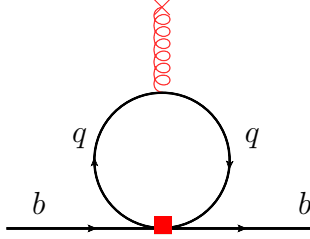


Figure 3.2: One-gluon corrections to the matrix elements of the dimension-six $\Delta B = 0$ four-quark operators. Through mixing into the Darwin operator, these contributions cancel the infrared divergences arising in the two-quark matching calculation.

in Ref. [81] for the generalised effective Hamiltonian in Eq. (3.1), and we make use of the corresponding results.

The situation is different for the $b \rightarrow u\bar{c}s$ and $b \rightarrow u\bar{u}d$ channels. In these cases, the PI and WE topologies alone are insufficient to remove the IR logarithms. The cancellation is completed only after including the WA topology, shown in the right panel of Figure 3.1. The corresponding WA contributions have not previously been determined for the generalised effective Hamiltonian in Eq. (3.1). We therefore compute these contributions in this work. Since the calculation follows the same methodology as that presented in Section 4.2 (see also [62]), differing only in the underlying operator basis, we do not repeat the technical details here and instead quote the final matching coefficients.

As discussed in the Chapter 2, the one-loop diagram shown in Figure 3.2 is both UV and IR divergent. The UV divergences are regulated using dimensional regularisation and renormalised in the $\overline{\text{MS}}$ scheme. To regulate the IR divergences, we employ the same prescription as in the matching calculation of the two-quark operators, namely by keeping the light quark propagating in the loop massive. This gives rise to logarithms of the form $\ln(m_q^2/m_b^2)$, which cancel exactly against the corresponding logarithms in the bare two-quark matching coefficients.

The renormalisation of the generalised $\Delta B = 0$ operator basis in Eqs. (3.12) to (3.15) is particularly simple. Only $\mathcal{O}_{1-4}$ and the corresponding chirality-flipped operators $\mathcal{O}'_{1-4}$ mix into the Darwin operator at one loop. As these operators already occur in the SM, their renormalised matrix elements are unchanged and can be taken directly from the SM calculations. At the renormalisation scale $\mu = m_b$ they are given by

$$\begin{aligned} \langle \mathcal{O}_{1,2} \rangle^{\text{ren}} &= \langle \mathcal{O}'_{1,2} \rangle^{\text{ren}} = \frac{a_{1,2}}{12\pi^2} \left[\log\left(\frac{m_b^2}{m_q^2}\right) + b_{1,2} \right] \langle \mathcal{O}_{\rho_D} \rangle + \mathcal{O}\left(\frac{1}{m_b}\right), \\ \langle \mathcal{O}_{3,4}^{(\prime)} \rangle^{\text{ren}} &= -\frac{1}{2N_c} \langle \mathcal{O}_{1,2}^{(\prime)} \rangle^{\text{ren}}, \end{aligned} \quad (3.16)$$

The numerical coefficients are given by $a_1 = 2$, $a_2 = -1$, $b_1 = -1$, and $b_2 = 0$, in

agreement with the corresponding SM results [62, 63].³ The remaining operators $\mathcal{O}_{5-8}$ and their chirality-flipped counterparts $\mathcal{O}'_{5-8}$ do not contribute to the Darwin operator through the one-loop diagrams of Figure 3.2, as their renormalised matrix elements vanish identically. Nevertheless, these operators are not irrelevant at dimension six: they contribute at tree level through the WA topology.

3.3 Analytical Results

Having established the generalised effective Hamiltonian, the matching procedure, and the renormalisation of the relevant operators, we are now in a position to present the corresponding HQE matching coefficients. Throughout this section, we consider the two-quark contributions to the inclusive non-leptonic decay $b \rightarrow q_1 \bar{q}_2 q_3$ up to and including dimension six.

The decay width can be written in the compact form

$$\Gamma(b \rightarrow q_1 \bar{q}_2 q_3) = \Gamma_0 |V_{q_1 b}|^2 |V_{q_2 q_3}|^2 \sum_{i,j=1}^{20} \mathcal{C}_i \mathcal{C}_j^* \mathcal{N}_{ij} \mathcal{F}_{ij}^{(q_1 q_2 q_3)}(\rho) + \mathcal{O}(1/m_b^4), \quad (3.17)$$

where

$$\Gamma_0 \equiv \frac{G_F^2 m_b^5}{192 \pi^3}, \quad (3.18)$$

and, for convenience, we relabel the primed Wilson coefficients according to $\mathcal{C}_{11} \equiv \mathcal{C}'_1, \dots, \mathcal{C}_{20} \equiv \mathcal{C}'_{10}$. The coefficient functions $\mathcal{F}_{ij}^{(q_1 q_2 q_3)}(\rho)$ encode the short-distance dynamics associated with the operator product $\mathcal{Q}_i^{q_1 q_2 q_3} \mathcal{Q}_j^{\dagger q_1 q_2 q_3}$. Their HQE expansion is parameterised as

$$\mathcal{F}_{ij}^{(q_1 q_2 q_3)} = \left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) P_{ij}^{(q_1 q_2 q_3)}(\rho) + \frac{\mu_G^2}{m_b^2} G_{ij}^{(q_1 q_2 q_3)}(\rho) + \frac{\rho_D^3}{m_b^3} D_{ij}^{(q_1 q_2 q_3)}(\rho), \quad (3.19)$$

where $\rho = m_c^2/m_b^2$. The coefficient of the kinetic-energy operator μ_π^2 has been fixed using reparametrisation invariance [88], implying that it is identical to the leading-power coefficient up to the universal factor $-1/2$. The colour factors $\mathcal{N}_{ij}$ entering Eq. (3.17) are

$$\mathcal{N}_{ij} = \begin{cases} 1, & \text{if } j - i \text{ odd} \\ N_c, & \text{if } j - i \text{ even} \end{cases}. \quad (3.20)$$

The coefficient functions $\mathcal{F}_{i,j}^{(q_1 q_2 q_3)}(\rho)$ satisfy the universal relations

$$\mathcal{F}_{i,j}^{(q_1 q_2 q_3)} = \mathcal{F}_{j,i}^{(q_1 q_2 q_3)}, \quad \mathcal{F}_{i+10,j+10}^{(q_1 q_2 q_3)} = \mathcal{F}_{i,j}^{(q_1 q_2 q_3)}, \quad (3.21)$$

$$\mathcal{F}_{i,j}^{(q_1 q_2 q_3)}(\rho) = 0, \quad 1 \leq i \leq 10, \quad 11 \leq j \leq 20, \quad (3.22)$$

which hold for all operator combinations. Consequently, it is sufficient to present only the independent coefficient functions $\mathcal{F}_{i,j}^{(q_1 q_2 q_3)}(\rho)$ with $1 \leq i \leq j \leq 10$.

³The numerical values of these coefficients are basis dependent. In particular, Ref. [63] employs a $\Delta B = 0$ operator basis constructed from colour-rearranged operators rather than the colour-octet basis adopted here, leading to different values for these constants.

The calculation is performed for all inclusive non-leptonic b -quark decay channels by considering the distinct choices of massive and massless final-state quarks listed in [Table 3.1](#). To avoid interrupting the presentation with lengthy analytical expressions,

Mass configuration	Corresponding transitions
$m_1 \neq 0, m_2 = m_3 = 0$	$b \rightarrow c\bar{u}d, b \rightarrow c\bar{u}s$
$m_1 = m_2 \neq 0, m_3 = 0$	$b \rightarrow c\bar{c}s, b \rightarrow c\bar{c}d$
$m_2 \neq 0, m_1 = m_3 = 0$	$b \rightarrow u\bar{c}s, b \rightarrow u\bar{c}d$
$m_1 = m_2 = m_3 = 0$	$b \rightarrow u\bar{u}d, b \rightarrow u\bar{u}s$

Table 3.1: Mass configurations considered in the matching calculation.

we restrict the discussion below to the CKM-favoured transitions $b \rightarrow c\bar{u}d$ and $b \rightarrow c\bar{c}s$. The analytical results for the additional decay channels are collected in [Appendix C](#); the complete set of results is also available in Ref. [89] and the accompanying ancillary files.

As discussed in [Section 3.1](#), the operator identities in [Eqs. \(3.8\) to \(3.10\)](#) relate the generalised operator basis to the SM QCD-penguin operators through Fierz transformations. Consequently, once the matching coefficients $P_{ij}^{(q_1 q_2 q_3)}(\rho)$, $G_{ij}^{(q_1 q_2 q_3)}(\rho)$, and $D_{ij}^{(q_1 q_2 q_3)}(\rho)$ have been determined, the corresponding SM QCD-penguin contributions follow immediately. Since the light-quark masses are neglected, only the mass configurations with $m_1 = m_2 = m_c$ and $m_1 = m_2 = m_3 = 0$ are relevant. While the leading-power penguin contributions were already known [90, 91], the dimension-five and dimension-six corrections presented here constitute new results and are discussed in [Section C.4](#).

3.3.1 Results for the $b \rightarrow c\bar{u}d$ Transition

Dimension-three contribution (Γ_3): We begin with the leading-power contribution to the transition $b \rightarrow c\bar{u}d$. Among the coefficient functions $P_{ij}^{(cud)}(\rho)$ defined in [Eq. \(3.19\)](#), only two are independent. They are given by

$$P_{1,1}^{(cud)}(\rho) = 1 - 8\rho - 12\rho^2 \log \rho + 8\rho^3 - \rho^4, \quad (3.23)$$

$$P_{1,3}^{(cud)}(\rho) = -2\sqrt{\rho} (1 + 9\rho + 6\rho(1 + \rho) \log \rho - 9\rho^2 - \rho^3). \quad (3.24)$$

The remaining non-vanishing coefficient functions are obtained from the following symmetry and proportionality relations:

$$\begin{aligned} P_{2,2}^{(cud)} &= P_{3,3}^{(cud)} = P_{4,4}^{(cud)} = P_{1,1}^{(cud)}, & P_{5,5}^{(cud)} &= P_{6,6}^{(cud)} = P_{7,7}^{(cud)} = P_{8,8}^{(cud)} = \frac{1}{4}P_{1,1}^{(cud)}, \\ P_{9,9}^{(cud)} &= P_{10,10}^{(cud)} = 12 P_{1,1}^{(cud)}, & P_{2,4}^{(cud)} &= P_{1,3}^{(cud)}, & P_{5,7}^{(cud)} &= P_{6,8}^{(cud)} = -\frac{1}{2}P_{1,3}^{(cud)}, \\ P_{i,j}^{(cud)} &= P_{i,j-1}^{(cud)} \quad \text{for odd } i \text{ and even } j, \end{aligned} \quad (3.25)$$

All other coefficient functions vanish identically.⁴

Dimension-five contribution (Γ_5): In view of the relation in [Eq. \(3.19\)](#), only the chromomagnetic matching coefficients need to be given explicitly. The independent non-vanishing functions $G_{i,j}^{(cud)}(\rho)$ are

$$G_{1,1}^{(cud)}(\rho) = -\frac{1}{2}(5\rho^4 - 24\rho^3 + 24\rho^2 + 12\rho^2 \log \rho - 8\rho + 3), \quad (3.26)$$

$$G_{1,2}^{(cud)}(\rho) = -\frac{1}{2}(5\rho^4 - 40\rho^3 + 72\rho^2 + 12\rho^2 \log \rho - 56\rho + 19), \quad (3.27)$$

$$G_{1,3}^{(cud)}(\rho) = \frac{1}{3}\sqrt{\rho}(13\rho^3 - 27\rho^2 - 6(3\rho^2 - 3\rho + 2) \log \rho + 27\rho - 13), \quad (3.28)$$

$$G_{2,4}^{(cud)}(\rho) = \sqrt{\rho}(5\rho^3 - 3\rho^2 - 6(1 + \rho)\rho \log \rho + 3\rho - 5), \quad (3.29)$$

$$G_{4,4}^{(cud)}(\rho) = -\frac{1}{2}(5\rho^4 - 8\rho^3 - 24\rho^2 + 12\rho^2 \log \rho + 40\rho - 13), \quad (3.30)$$

$$G_{5,5}^{(cud)}(\rho) = -\frac{1}{8}(5\rho^4 - 32\rho^3 + 72\rho^2 + 12(\rho - 4)\rho \log \rho - 32\rho - 13), \quad (3.31)$$

$$G_{5,7}^{(cud)}(\rho) = -\frac{3}{2}\sqrt{\rho}(\rho^3 - 3\rho^2 - 2(\rho^2 - 5\rho - 2) \log \rho - 9\rho + 11), \quad (3.32)$$

$$G_{6,8}^{(cud)}(\rho) = \frac{3}{2}(1 - \rho)\sqrt{\rho}(\rho^2 - 2\rho \log \rho - 1), \quad (3.33)$$

$$G_{6,10}^{(cud)}(\rho) = -4\sqrt{\rho}(\rho^3 - 6\rho \log \rho + 3\rho - 4), \quad (3.34)$$

$$G_{8,10}^{(cud)}(\rho) = 8(1 - \rho)^3, \quad (3.35)$$

$$G_{9,9}^{(cud)}(\rho) = -2(15\rho^4 - 64\rho^3 + 24\rho^2 + 12(3\rho + 4)\rho \log \rho + 25), \quad (3.36)$$

$$G_{10,10}^{(cud)}(\rho) = -2(15\rho^4 - 40\rho^3 - 24\rho^2 + 36\rho^2 \log \rho + 72\rho - 23). \quad (3.37)$$

The remaining non-vanishing coefficient functions are obtained from the relations

$$\begin{aligned} G_{3,3}^{(cud)} &= G_{2,2}^{(cud)} = G_{1,1}^{(cud)}, & G_{7,7}^{(cud)} &= G_{5,5}^{(cud)}, & G_{8,8}^{(cud)} &= G_{6,6}^{(cud)} = \frac{1}{4}G_{1,1}^{(cud)}, \\ G_{3,2}^{(cud)} &= G_{1,4}^{(cud)} = G_{1,3}^{(cud)}, & G_{3,4}^{(cud)} &= G_{4,4}^{(cud)}, & G_{5,6}^{(cud)} &= G_{5,5}^{(cud)}, \\ G_{5,8}^{(cud)} &= G_{5,7}^{(cud)}, & G_{5,10}^{(cud)} &= 2G_{6,10}^{(cud)}, & G_{9,6}^{(cud)} &= 2G_{6,10}^{(cud)}, \\ G_{7,8}^{(cud)} &= G_{7,7}^{(cud)}, & G_{7,10}^{(cud)} &= 2G_{8,10}^{(cud)}, & G_{7,6}^{(cud)} &= G_{5,7}^{(cud)}, \\ G_{9,8}^{(cud)} &= G_{7,10}^{(cud)}, & G_{9,10}^{(cud)} &= G_{9,9}^{(cud)}, \end{aligned} \quad (3.38)$$

while all other coefficient functions vanish identically.

⁴After translating between the two operator bases and setting $N_c = 3$, we find complete agreement with the dimension-three results of Ref. [\[86\]](#).

Dimension-six contribution (Γ_6): For the Darwin operator, the independent non-vanishing coefficient functions $D_{i,j}^{(cud)}(\rho)$ in Eq. (3.19) are

$$D_{1,1}^{(cud)}(\rho) = -\frac{2}{3}(5\rho^4 - 16\rho^3 + 12\rho^2 + 16\rho - 17 - 12\log\rho), \quad (3.39)$$

$$D_{1,2}^{(cud)}(\rho) = -\frac{2}{3}(5\rho^4 - 54\rho^3 + 90\rho^2 - 50\rho + 9) - 16(\rho - 1)^3 \log(1 - \rho) + 8(\rho^3 - 3\rho^2 + 1) \log\rho, \quad (3.40)$$

$$D_{1,3}^{(cud)}(\rho) = \frac{8}{3}\sqrt{\rho}(2\rho^3 - 9\rho^2 + 18\rho - 11 - 6\log\rho), \quad (3.41)$$

$$D_{1,4}^{(cud)}(\rho) = \frac{16}{3}\sqrt{\rho}(\rho^3 - 6\rho^2 + 12\rho - 7 - 3\log\rho), \quad (3.42)$$

$$D_{2,2}^{(cud)}(\rho) = \frac{2}{3}(1 - \rho)(5\rho^3 - 25\rho^2 + 11\rho + 9 + 24(\rho^2 - 1)\log(1 - \rho) - 12\rho^2 \log\rho), \quad (3.43)$$

$$D_{2,4}^{(cud)}(\rho) = \frac{4}{3}\sqrt{\rho}(5\rho^3 - 24\rho^2 + 27\rho - 8 + 12(\rho - 1)^2 \log(1 - \rho) - 6\rho(\rho - 2) \log\rho), \quad (3.44)$$

$$D_{3,4}^{(cud)}(\rho) = -\frac{2}{3}(5\rho^4 + 18\rho^3 - 78\rho^2 + 118\rho - 63 + 24(1 - \rho)^3 \log(1 - \rho) + 12(\rho^3 - 3\rho^2 - 1) \log\rho), \quad (3.45)$$

$$D_{4,4}^{(cud)}(\rho) = -\frac{2}{3}(5\rho^4 + 8\rho^3 - 66\rho^2 + 112\rho - 59 + 48(1 - \rho)^2 \log(1 - \rho) - 24\rho^2 \log\rho), \quad (3.46)$$

$$D_{5,5}^{(cud)}(\rho) = \frac{1}{6}(-5\rho^4 + 22\rho^3 - 54\rho^2 - 22\rho + 59 + 12(5\rho + 2) \log\rho), \quad (3.47)$$

$$D_{5,6}^{(cud)}(\rho) = \frac{1}{6}(-5\rho^4 + 26\rho^3 - 66\rho^2 - 10\rho + 55 + 12(5\rho + 2) \log\rho), \quad (3.48)$$

$$D_{6,8}^{(cud)}(\rho) = -\frac{4}{3}\sqrt{\rho}(\rho^3 - 6\rho^2 + 6\rho - 1 + 6(\rho - 1)^2 \log(1 - \rho) - 3\rho(\rho - 2) \log\rho), \quad (3.49)$$

$$D_{6,10}^{(cud)}(\rho) = -8\sqrt{\rho}(\rho - 2)(\rho - 1)^2, \quad (3.50)$$

$$D_{7,8}^{(cud)}(\rho) = \frac{1}{6}(-5\rho^4 + 18\rho^3 - 42\rho^2 - 34\rho + 63) + 2(5\rho + 2) \log\rho, \quad (3.51)$$

$$D_{7,10}^{(cud)}(\rho) = 16(-3\rho^3 + 7\rho^2 - 7\rho + 3 + 2(\rho - 1)^3 \log(1 - \rho) - \rho^2(\rho - 3) \log\rho), \quad (3.52)$$

$$D_{8,8}^{(cud)}(\rho) = \frac{1}{6}(1 - \rho)(5\rho^3 - 21\rho^2 + 3\rho + 13 + 24(\rho^2 - 1) \log(1 - \rho) - 12\rho^2 \log\rho), \quad (3.53)$$

$$D_{8,10}^{(cud)}(\rho) = \frac{4}{3}(-17\rho^3 + 45\rho^2 - 51\rho + 23 + 12(\rho - 1)^3 \log(1 - \rho) - 6\rho^2(\rho - 3) \log\rho), \quad (3.54)$$

$$D_{9,9}^{(cud)}(\rho) = -8(5\rho^4 - 14\rho^3 - 2\rho^2 + 14\rho - 3 + 4(5\rho - 2) \log\rho), \quad (3.55)$$

$$D_{9,10}^{(cud)}(\rho) = -\frac{8}{3}(15\rho^4 - 38\rho^3 - 66\rho^2 + 150\rho - 61 + 12(5\rho - 2)\log\rho), \quad (3.56)$$

$$D_{10,10}^{(cud)}(\rho) = -\frac{8}{3}(15\rho^4 - 10\rho^3 - 108\rho^2 + 234\rho - 131 + 24(\rho - 1)^2(\rho + 5)\log(1 - \rho) - 12\rho^2(\rho + 3)\log\rho). \quad (3.57)$$

The remaining non-vanishing coefficient functions are obtained from

$$\begin{aligned} D_{3,3}^{(cud)} &= D_{1,1}^{(cud)}, & D_{7,7}^{(cud)} &= D_{5,5}^{(cud)}, & D_{6,6}^{(cud)} &= \frac{1}{4}D_{2,2}^{(cud)}, \\ D_{5,7}^{(cud)} &= -\frac{1}{4}D_{1,3}^{(cud)}, & D_{5,8}^{(cud)} &= -\frac{1}{4}D_{1,3}^{(cud)}, & D_{5,10}^{(cud)} &= \frac{2(\rho - 1)}{\rho - 2}D_{6,10}^{(cud)}, \\ D_{i,j}^{(cud)} &= D_{i-1,j+1}^{(cud)} & \text{for even } i \text{ and odd } j, \end{aligned} \quad (3.58)$$

while all other coefficient functions vanish identically.

3.3.2 Results for the $b \rightarrow c\bar{c}s$ Transition

Dimension-three contribution (Γ_3): For convenience, we define

$$\Delta_\rho \equiv \sqrt{1 - 4\rho}, \quad \Lambda_\rho \equiv \log\left(\frac{1 + \Delta_\rho}{1 - \Delta_\rho}\right). \quad (3.59)$$

The independent non-vanishing coefficient functions $P_{i,j}^{(ccs)}(\rho)$ are

$$P_{1,1}^{(ccs)}(\rho) = \Delta_\rho (1 - 14\rho - 2\rho^2 - 12\rho^3) + 24\rho^2(1 - \rho^2)\Lambda_\rho, \quad (3.60)$$

$$P_{1,3}^{(ccs)}(\rho) = 2\sqrt{\rho} [\Delta_\rho (-1 - 5\rho + 6\rho^2) + 6\rho(1 - 2\rho + 2\rho^2)\Lambda_\rho]. \quad (3.61)$$

The remaining non-vanishing coefficient functions are obtained from

$$\begin{aligned} P_{2,2}^{(ccs)} &= P_{3,3}^{(ccs)} = P_{4,4}^{(ccs)} = P_{1,1}^{(ccs)}, \\ P_{5,5}^{(ccs)} &= P_{6,6}^{(ccs)} = P_{7,7}^{(ccs)} = P_{8,8}^{(ccs)} = \frac{1}{4}P_{1,1}^{(ccs)}, \\ P_{9,9}^{(ccs)} &= P_{10,10}^{(ccs)} = 12P_{1,1}^{(ccs)}, \\ P_{2,4}^{(ccs)} &= P_{1,3}^{(ccs)}, & P_{5,7}^{(ccs)} &= P_{6,8}^{(ccs)} = -\frac{1}{2}P_{1,3}^{(ccs)}, \\ P_{1,5}^{(ccs)} &= P_{3,7}^{(ccs)} = P_{2,6}^{(ccs)} = P_{4,8}^{(ccs)} = \sqrt{\rho}P_{1,3}^{(ccs)}, \\ P_{1,7}^{(ccs)} &= P_{3,5}^{(ccs)} = P_{2,8}^{(ccs)} = P_{4,6}^{(ccs)} = \frac{1}{2}P_{1,3}^{(ccs)}, \\ P_{1,9}^{(ccs)} &= P_{2,10}^{(ccs)} = -6P_{1,3}^{(ccs)}, & P_{3,9}^{(ccs)} &= P_{4,10}^{(ccs)} = 12\sqrt{\rho}P_{1,3}^{(ccs)}, \\ P_{i,j}^{(ccs)} &= P_{i,j-1}^{(ccs)}, & \text{for odd } i \text{ and even } j. \end{aligned} \quad (3.62)$$

while all other coefficient functions vanish identically.

Dimension-five contribution (Γ_5): The kinetic contribution is obtained from the leading-power result according to Eq. (3.19). The independent non-vanishing chromomagnetic coefficient functions $G_{i,j}^{(ccs)}(\rho)$ are

$$G_{1,1}^{(ccs)}(\rho) = -\frac{1}{2\Delta_\rho} \left[3 - 22\rho + 50\rho^2 + 20\rho^3 - 240\rho^4 + 24\Delta_\rho\rho^2(-1 + 5\rho^2)\Lambda_\rho \right], \quad (3.63)$$

$$G_{1,2}^{(ccs)}(\rho) = \frac{1}{2\Delta_\rho} \left[-19 + 78\rho - 66\rho^2 + 172\rho^3 + 240\rho^4 - 24\Delta_\rho\rho(-2 - \rho + 4\rho^2 + 5\rho^3)\Lambda_\rho \right], \quad (3.64)$$

$$G_{1,3}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{3\Delta_\rho} \left[-13 + 23\rho + 266\rho^2 - 600\rho^3 + 6\Delta_\rho(2 - 3\rho - 18\rho^2 + 50\rho^3)\Lambda_\rho \right], \quad (3.65)$$

$$G_{1,4}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{3\Delta_\rho} \left[-13 + 41\rho + 302\rho^2 - 1032\rho^3 + 6\Delta_\rho(2 - 9\rho - 18\rho^2 + 86\rho^3)\Lambda_\rho \right], \quad (3.66)$$

$$G_{1,5}^{(ccs)}(\rho) = \frac{\rho}{\Delta_\rho} \left[10 - 71\rho + 154\rho^2 - 120\rho^3 + 6\Delta_\rho(-1 + 5\rho - 12\rho^2 + 10\rho^3)\Lambda_\rho \right], \quad (3.67)$$

$$G_{1,6}^{(ccs)}(\rho) = \frac{\rho}{\Delta_\rho} \left[7 - 53\rho + 130\rho^2 - 120\rho^3 + 6\Delta_\rho\rho(1 - 10\rho + 10\rho^2)\Lambda_\rho \right], \quad (3.68)$$

$$G_{1,7}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{2\Delta_\rho} \left[-5 + 13\rho + 22\rho^2 + 24\rho^3 - 6\Delta_\rho\rho(-3 + 2\rho + 2\rho^2)\Lambda_\rho \right], \quad (3.69)$$

$$G_{1,8}^{(ccs)}(\rho) = -\frac{3\sqrt{\rho}}{2\Delta_\rho} \left[9 - 37\rho + 10\rho^2 - 24\rho^3 + 2\Delta_\rho(-2 - \rho - 2\rho^2 + 6\rho^3)\Lambda_\rho \right], \quad (3.70)$$

$$G_{1,9}^{(ccs)}(\rho) = \frac{2\sqrt{\rho}}{\Delta_\rho} \left[-17 + 73\rho - 98\rho^2 + 312\rho^3 - 6\Delta_\rho\rho(-7 - 6\rho + 26\rho^2)\Lambda_\rho \right], \quad (3.71)$$

$$G_{1,10}^{(ccs)}(\rho) = -\frac{6\sqrt{\rho}}{\Delta_\rho} \left[-27 + 119\rho - 14\rho^2 - 120\rho^3 + 2\Delta_\rho(6 - 5\rho + 6\rho^2 + 30\rho^3)\Lambda_\rho \right], \quad (3.72)$$

$$G_{2,4}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{\Delta_\rho} \left[-5 + 19\rho + 34\rho^2 - 120\rho^3 + 6\Delta_\rho\rho(1 - 2\rho + 10\rho^2)\Lambda_\rho \right], \quad (3.73)$$

$$G_{2,8}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{2\Delta_\rho} \left[11 - 49\rho + 26\rho^2 - 24\rho^3 + 6\Delta_\rho\rho(-3 - 2\rho + 2\rho^2)\Lambda_\rho \right], \quad (3.74)$$

$$G_{2,10}^{(ccs)}(\rho) = \frac{2\sqrt{\rho}}{\Delta_\rho} \left[-1 + 11\rho - 94\rho^2 + 264\rho^3 + 6\Delta_\rho\rho(1 + 6\rho - 22\rho^2)\Lambda_\rho \right], \quad (3.75)$$

$$G_{3,4}^{(ccs)}(\rho) = \frac{1}{2\Delta_\rho} \left[13 - 106\rho + 206\rho^2 - 20\rho^3 + 240\rho^4 - 24\Delta_\rho\rho^2(-1 + 5\rho^2)\Lambda_\rho \right], \quad (3.76)$$

$$G_{3,5}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{2\Delta_\rho} \left[-5 + 13\rho + 22\rho^2 + 24\rho^3 - 6\Delta_\rho\rho(-3 + 2\rho + 2\rho^2)\Lambda_\rho \right], \quad (3.77)$$

$$G_{3,6}^{(ccs)}(\rho) = \frac{\sqrt{\rho}}{6\Delta_\rho} \left[-17 + 73\rho - 98\rho^2 + 312\rho^3 - 6\Delta_\rho(2 - 15\rho - 6\rho^2 + 26\rho^3)\Lambda_\rho \right], \quad (3.78)$$

$$G_{3,7}^{(ccs)}(\rho) = \frac{\rho}{\Delta_\rho} \left[10 - 71\rho + 154\rho^2 - 120\rho^3 + 6\Delta_\rho(-1 + 5\rho - 12\rho^2 + 10\rho^3)\Lambda_\rho \right], \quad (3.79)$$

$$G_{3,8}^{(ccs)}(\rho) = \frac{\rho}{\Delta_\rho} \left[13 - 89\rho + 178\rho^2 - 120\rho^3 + 6\Delta_\rho(-2 + 9\rho - 14\rho^2 + 10\rho^3)\Lambda_\rho \right], \quad (3.80)$$

$$G_{3,9}^{(ccs)}(\rho) = \frac{12\rho}{\Delta_\rho} \left[-2 + \rho + 58\rho^2 - 120\rho^3 + 2\Delta_\rho(1 - \rho - 12\rho^2 + 30\rho^3)\Lambda_\rho \right], \quad (3.81)$$

$$G_{3,10}^{(ccs)}(\rho) = \frac{12\rho}{\Delta_\rho} \left[-11 + 55\rho - 14\rho^2 - 120\rho^3 + 2\Delta_\rho(2 - 5\rho + 6\rho^2 + 30\rho^3)\Lambda_\rho \right], \quad (3.82)$$

$$G_{5,5}^{(ccs)}(\rho) = \frac{1}{8\Delta_\rho} \left[13 - 34\rho - 34\rho^2 - 212\rho^3 + 240\rho^4 - 24\Delta_\rho\rho(2 - \rho - 4\rho^2 + 5\rho^3)\Lambda_\rho \right], \quad (3.83)$$

$$G_{5,7}^{(ccs)}(\rho) = -\frac{3\sqrt{\rho}}{2\Delta_\rho} \left[11 - 49\rho + 26\rho^2 - 24\rho^3 + 2\Delta_\rho(-2 - \rho - 6\rho^2 + 6\rho^3)\Lambda_\rho \right], \quad (3.84)$$

$$G_{5,10}^{(ccs)}(\rho) = 4\sqrt{\rho} \left[\Delta_\rho(8 + \rho + 6\rho^2) + 6\rho(-3 + 2\rho^2)\Lambda_\rho \right], \quad (3.85)$$

$$G_{6,8}^{(ccs)}(\rho) = \frac{3\sqrt{\rho}}{2\Delta_\rho} \left[-1 + 5\rho - 10\rho^2 + 24\rho^3 + 2\Delta_\rho\rho(1 + 2\rho - 6\rho^2)\Lambda_\rho \right], \quad (3.86)$$

$$G_{6,10}^{(ccs)}(\rho) = -4\sqrt{\rho} \left[\Delta_\rho(-4 + \rho + 6\rho^2) + 6(\rho + 2\rho^3)\Lambda_\rho \right], \quad (3.87)$$

$$G_{7,10}^{(ccs)}(\rho) = 4 \left[\Delta_\rho(4 - 7\rho + 6\rho^2) + 6\rho(-1 + 2\rho^2)\Lambda_\rho \right], \quad (3.88)$$

$$G_{8,10}^{(ccs)}(\rho) = 8\Delta_\rho^3, \quad (3.89)$$

$$G_{9,9}^{(ccs)}(\rho) = \frac{2}{\Delta_\rho} \left[-25 + 122\rho - 166\rho^2 + 132\rho^3 + 720\rho^4 - 24\Delta_\rho\rho(-2 - 3\rho + 4\rho^2 + 15\rho^3)\Lambda_\rho \right], \quad (3.90)$$

$$G_{9,10}^{(ccs)}(\rho) = \frac{2}{\Delta_\rho} \left[-25 - 22\rho + 314\rho^2 + 516\rho^3 + 720\rho^4 - 72\Delta_\rho\rho(-2 - \rho + 4\rho^2 + 5\rho^3)\Lambda_\rho \right], \quad (3.91)$$

$$G_{10,10}^{(ccs)}(\rho) = \frac{2}{\Delta_\rho} \left[23 - 190\rho + 362\rho^2 - 60\rho^3 + 720\rho^4 - 72\Delta_\rho\rho^2(-1 + 5\rho^2)\Lambda_\rho \right]. \quad (3.92)$$

The remaining non-vanishing coefficient functions are obtained from

$$\begin{aligned}
G_{2,2}^{(ccs)} &= G_{1,1}^{(ccs)}, & G_{2,6}^{(ccs)} &= G_{1,6}^{(ccs)}, & G_{3,3}^{(ccs)} &= G_{1,1}^{(ccs)}, & G_{4,4}^{(ccs)} &= G_{3,4}^{(ccs)}, \\
G_{3,5}^{(ccs)} &= G_{1,7}^{(ccs)}, & G_{4,6}^{(ccs)} &= \frac{1}{2}G_{2,4}^{(ccs)}, & G_{4,8}^{(ccs)} &= \sqrt{\rho}G_{2,4}^{(ccs)}, & G_{4,10}^{(ccs)} &= 12\sqrt{\rho}G_{2,4}^{(ccs)}, \\
G_{5,6}^{(ccs)} &= G_{5,5}^{(ccs)}, & G_{5,8}^{(ccs)} &= G_{5,7}^{(ccs)}, & G_{6,6}^{(ccs)} &= \frac{1}{4}G_{3,3}^{(ccs)}, & G_{7,8}^{(ccs)} &= G_{7,7}^{(ccs)} = G_{5,5}^{(ccs)}, \\
G_{8,8}^{(ccs)} &= G_{6,6}^{(ccs)}, & G_{i,j}^{(ccs)} &= G_{i-1,j+1}^{(ccs)} & \text{for even } i \text{ and odd } j,
\end{aligned} \tag{3.93}$$

while all other coefficient functions vanish identically.

Dimension-six contribution (Γ_6): The independent non-vanishing Darwin coefficient functions $D_{i,j}^{(ccs)}(\rho)$ in Eq. (3.19) are

$$D_{1,1}^{(ccs)}(\rho) = -\frac{2}{3} \left[\Delta_\rho(-17 - 8\rho + 22\rho^2 + 60\rho^3) + 12(1 - \rho - 2\rho^2 + 2\rho^3 + 10\rho^4)\Lambda_\rho \right], \tag{3.94}$$

$$\begin{aligned}
D_{1,2}^{(ccs)}(\rho) &= -\frac{2}{3}\Delta_\rho \left[33 - 46\rho + 106\rho^2 + 60\rho^3 + 24\log(1 - 4\rho) - 24\log\rho \right] \\
&\quad + 8 \left[3 - 2\rho + 4\rho^2 - 16\rho^3 - 10\rho^4 \right] \Lambda_\rho
\end{aligned} \tag{3.95}$$

$$D_{1,3}^{(ccs)}(\rho) = \frac{8}{3}\sqrt{\rho} \left[\Delta_\rho(-11 - 7\rho + 30\rho^2) + 6(1 - 4\rho^2 + 10\rho^3)\Lambda_\rho \right], \tag{3.96}$$

$$\begin{aligned}
D_{1,4}^{(ccs)}(\rho) &= \frac{4}{3}\sqrt{\rho} \left[\Delta_\rho(-22 - 23\rho + 114\rho^2 + 12\log(1 - 4\rho) - 12\log(\rho)) \right. \\
&\quad \left. + 6(-1 + 6\rho - 14\rho^2 + 38\rho^3)\Lambda_\rho \right],
\end{aligned} \tag{3.97}$$

$$D_{1,5}^{(ccs)}(\rho) = \frac{4}{3}\rho \left[\Delta_\rho(7 - 31\rho + 30\rho^2) + 3(-1 + 8\rho - 24\rho^2 + 20\rho^3)\Lambda_\rho \right], \tag{3.98}$$

$$D_{1,7}^{(ccs)}(\rho) = -\frac{4}{3}\sqrt{\rho} \left[2\Delta_\rho(2 - 5\rho + 6\rho^2) + 3(-1 + 4\rho - 8\rho^2 + 8\rho^3)\Lambda_\rho \right], \tag{3.99}$$

$$\begin{aligned}
D_{1,8}^{(ccs)}(\rho) &= \frac{2}{3}\sqrt{\rho} \left[\Delta_\rho(7 + 29\rho - 42\rho^2 + 12\log(1 - 4\rho) - 12\log\rho) \right. \\
&\quad \left. - 12(1 + \rho - 6\rho^2 + 7\rho^3)\Lambda_\rho \right],
\end{aligned} \tag{3.100}$$

$$D_{1,9}^{(ccs)}(\rho) = -16\sqrt{\rho} \left[2\Delta_\rho(-1 + \rho + 6\rho^2) + 3(1 - 4\rho + 8\rho^3)\Lambda_\rho \right], \tag{3.101}$$

$$\begin{aligned}
D_{1,10}^{(ccs)}(\rho) &= 8\sqrt{\rho} \left[-\Delta_\rho(3 + 17\rho + 30\rho^2 + 12\log(1 - 4\rho) - 12\log\rho) \right. \\
&\quad \left. + 12(1 + \rho - 2\rho^2 - 5\rho^3)\Lambda_\rho \right],
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
D_{2,2}^{(ccs)}(\rho) &= \frac{2}{3}\Delta_\rho \left[9 + 22\rho - 34\rho^2 - 60\rho^3 - 24\log(1 - 4\rho) + 24\log\rho \right] \\
&\quad - 16 \left[-1 + 2\rho + \rho^2 + 2\rho^3 + 5\rho^4 \right] \Lambda_\rho,
\end{aligned} \tag{3.103}$$

$$D_{2,4}^{(ccs)}(\rho) = \frac{4}{3}\sqrt{\rho}\left[\Delta_\rho(-8 + 11\rho + 30\rho^2 + 12\log(1 - 4\rho) - 12\log\rho) + 12(-1 + \rho + \rho^2 + 5\rho^3)\Lambda_\rho\right], \quad (3.104)$$

$$D_{2,6}^{(ccs)}(\rho) = \frac{4}{3}\rho\left[\Delta_\rho(7 - 7\rho + 30\rho^2 + 12\log(1 - 4\rho) - 12\log\rho) + 12(-1 + \rho - 2\rho^2 + 5\rho^3)\Lambda_\rho\right], \quad (3.105)$$

$$D_{2,8}^{(ccs)}(\rho) = \frac{2}{3}\sqrt{\rho}\left[\Delta_\rho(4 + 5\rho - 6\rho^2 + 12\log(1 - 4\rho) - 12\log\rho) - 12(-1 + \rho)^2(1 + \rho)\Lambda_\rho\right], \quad (3.106)$$

$$D_{2,10}^{(ccs)}(\rho) = 8\sqrt{\rho}\left[\Delta_\rho(4 - 9\rho - 18\rho^2 - 12\log(1 - 4\rho) + 12\log\rho) - 12(-1 + \rho + \rho^2 + 3\rho^3)\Lambda_\rho\right], \quad (3.107)$$

$$D_{3,4}^{(ccs)}(\rho) = -\frac{2}{3}\Delta_\rho\left[-59 + 106\rho + 10\rho^2 + 60\rho^3 + 48(1 - 2\rho)\log(1 - 4\rho) + 48(-1 + 2\rho)\log\rho\right] + 16(2 - 8\rho + 2\rho^2 - 5\rho^4)\Lambda_\rho, \quad (3.108)$$

$$D_{3,6}^{(ccs)}(\rho) = 2\sqrt{\rho}\left[\Delta_\rho(2 + 15\rho - 18\rho^2 + 4\log(1 - 4\rho) - 4\log\rho) - 2(3 + 2\rho - 18\rho^2 + 18\rho^3)\Lambda_\rho\right], \quad (3.109)$$

$$D_{3,8}^{(ccs)}(\rho) = \frac{2}{3}\rho\left[\Delta_\rho(11 - 32\rho + 60\rho^2 + 24\log(1 - 4\rho) - 24\log\rho) + 6(-7 + 16\rho - 14\rho^2 + 20\rho^3)\Lambda_\rho\right], \quad (3.110)$$

$$D_{3,9}^{(ccs)}(\rho) = 16\rho\left[\Delta_\rho(-2 - 13\rho + 30\rho^2) + (3 - 36\rho^2 + 60\rho^3)\Lambda_\rho\right], \quad (3.111)$$

$$D_{3,10}^{(ccs)}(\rho) = 8\rho\left[\Delta_\rho(-25 + 40\rho + 60\rho^2 + 24\log(1 - 4\rho) - 24\log\rho) + 6(-3 + 10\rho^2 + 20\rho^3)\Lambda_\rho\right], \quad (3.112)$$

$$D_{5,5}^{(ccs)}(\rho) = \frac{1}{6}\left[-\Delta_\rho(-59 + 16\rho - 50\rho^2 + 60\rho^3) - 12(2 + \rho + 4\rho^2 - 10\rho^3 + 10\rho^4)\Lambda_\rho\right], \quad (3.113)$$

$$D_{5,6}^{(ccs)}(\rho) = \frac{1}{6}\Delta_\rho[51 - 2\rho + 38\rho^2 - 60\rho^3 - 24\log(1 - 4\rho) + 24\log\rho] - 2(-1 + 6\rho + 8\rho^2 - 8\rho^3 + 10\rho^4)\Lambda_\rho, \quad (3.114)$$

$$D_{5,7}^{(ccs)}(\rho) = -\frac{2}{3}\sqrt{\rho}\left[\Delta_\rho(-11 - 4\rho + 12\rho^2) + 6(1 - 2\rho^2 + 4\rho^3)\Lambda_\rho\right], \quad (3.115)$$

$$D_{5,8}^{(ccs)}(\rho) = \frac{8}{3}\sqrt{\rho}\left[\Delta_\rho(2 + \rho - 3\rho^2 - 3\log(1 - 4\rho) + 3\log\rho) + 3(1 - 3\rho + \rho^2 - 2\rho^3)\Lambda_\rho\right], \quad (3.116)$$

$$D_{5,10}^{(ccs)}(\rho) = 8\sqrt{\rho} \left[\Delta_\rho(3 - 5\rho + 6\rho^2) + 12(-1 + \rho)\rho^2\Lambda_\rho \right], \quad (3.117)$$

$$D_{6,8}^{(ccs)}(\rho) = \frac{4}{3}\sqrt{\rho} \left[\Delta_\rho(1 - 4\rho - 6\rho^2 - 6\log(1 - 4\rho) + 6\log \rho) - 6(-1 + \rho + \rho^2 + 2\rho^3)\Lambda_\rho \right], \quad (3.118)$$

$$D_{6,10}^{(ccs)}(\rho) = -8\sqrt{\rho} \left[\Delta_\rho(-2 + \rho + 6\rho^2) + 12\rho^3\Lambda_\rho \right], \quad (3.119)$$

$$D_{7,8}^{(ccs)}(\rho) = \frac{1}{6}\Delta_\rho \left[55 - 18\rho + 38\rho^2 - 60\rho^3 - 24\log(1 - 4\rho) + 24\log \rho \right] - 2(-1 + 6\rho + 8\rho^2 - 8\rho^3 + 10\rho^4)\Lambda_\rho, \quad (3.120)$$

$$D_{7,10}^{(ccs)}(\rho) = \frac{8}{3} \left[2\Delta_\rho(11 - 17\rho + 12\rho^2 + 3(-1 + 4\rho)\log(1 - 4\rho) + 3(1 - 4\rho)\log \rho) + (3 - 42\rho + 48\rho^3)\Lambda_\rho \right], \quad (3.121)$$

$$D_{8,8}^{(ccs)}(\rho) = \frac{1}{6}\Delta_\rho \left[13 + 6\rho - 34\rho^2 - 60\rho^3 - 24\log(1 - 4\rho) + 24\log \rho \right] - 4[-1 + 2\rho + \rho^2 + 2\rho^3 + 5\rho^4]\Lambda_\rho, \quad (3.122)$$

$$D_{8,10}^{(ccs)}(\rho) = \frac{4}{3}\Delta_\rho \left[23 - 56\rho + 12\rho^2 + 12(-1 + 4\rho)\log(1 - 4\rho) + 12(1 - 4\rho)\log \rho \right] + 16(1 - 6\rho + 3\rho^2 + 2\rho^3)\Lambda_\rho, \quad (3.123)$$

$$D_{9,9}^{(ccs)}(\rho) = -8 \left[\Delta_\rho(-3 - 16\rho + 46\rho^2 + 60\rho^3) + 4(2 - 5\rho - 12\rho^2 + 18\rho^3 + 30\rho^4)\Lambda_\rho \right], \quad (3.124)$$

$$D_{9,10}^{(ccs)}(\rho) = -\frac{8}{3}\Delta_\rho \left[-5 + 134\rho + 270\rho^2 + 180\rho^3 - 24(-5 + 8\rho)\log(1 - 4\rho) + 24(-5 + 8\rho)\log \rho \right] - 32(-13 + 30\rho - 24\rho^2 + 40\rho^3 + 30\rho^4)\Lambda_\rho, \quad (3.125)$$

$$D_{10,10}^{(ccs)}(\rho) = -\frac{8}{3}\Delta_\rho \left[-131 + 206\rho + 54\rho^2 + 180\rho^3 - 24(-5 + 8\rho)\log(1 - 4\rho) + 24(-5 + 8\rho)\log \rho \right] - 64[-5 + 18\rho - 3\rho^2 + 2\rho^3 + 15\rho^4]\Lambda_\rho. \quad (3.126)$$

The remaining non-vanishing coefficient functions are obtained from the relations

$$\begin{aligned} D_{3,3}^{(ccs)} &= D_{1,1}^{(ccs)}, & D_{4,4}^{(ccs)} &= D_{3,4}^{(ccs)}, & D_{3,5}^{(ccs)} &= D_{1,7}^{(ccs)}, & D_{3,7}^{(ccs)} &= D_{1,5}^{(ccs)}, \\ D_{6,6}^{(ccs)} &= \frac{1}{4}D_{2,2}^{(ccs)}, & D_{4,6}^{(ccs)} &= \frac{1}{2}D_{2,4}^{(ccs)}, & D_{4,8}^{(ccs)} &= \sqrt{\rho}D_{2,4}^{(ccs)}, \\ D_{4,10}^{(ccs)} &= 12\sqrt{\rho}D_{2,4}^{(ccs)}, & D_{2,6}^{(ccs)} &= D_{1,6}^{(ccs)}, & D_{7,7}^{(ccs)} &= D_{5,5}^{(ccs)}, \\ D_{i,j}^{(ccs)} &= D_{i-1,j+1}^{(ccs)} & \text{for even } i \text{ and odd } j, \end{aligned} \quad (3.127)$$

while all other coefficient functions vanish identically.

Before concluding this chapter, we note that, although the calculations presented here were performed predominantly in four dimensions, the complete set of results

has been independently reproduced using dimensional regularisation in a separate calculation. The agreement between the two calculations provides a highly non-trivial validation of the analytical expressions obtained here. Details of the independent calculation can be found in Ref. [89].

3.4 Closing Remarks

The results presented in this chapter provide the two-quark HQE matching coefficients through dimension six for the most general dimension-six $|\Delta B| = 1$ four-fermion Hamiltonian considered in Eq. (3.1). They can therefore be used to study generic short-distance modifications of non-leptonic b -quark decays in a model-independent way and to constrain the corresponding Wilson coefficients using inclusive decay widths and lifetime ratios. Since all relevant flavour combinations $b \rightarrow q_1 \bar{q}_2 q_3$, with $q_1, q_2 = u, c$ and $q_3 = d, s$, have been considered, the results can be applied simultaneously to CKM-favoured and CKM-suppressed decay channels. In addition, the calculation of the previously missing WA topology completes the dimension-six four-quark matching contribution $\tilde{\Gamma}_6$ relevant to the present analysis.

An important consequence of the general analysis is the completion of the dimension-six two-quark contribution in the SM at leading order in QCD. In addition to the previously known current–current contributions, the results of Section C.4 provide the contributions induced by the QCD-penguin operators to the chromomagnetic and Darwin coefficients at dimensions five and six, respectively, while reproducing the known leading-power result. Together with the current–current sector, these results therefore complete the SM two-quark matching through dimension six at this order.

The generalised operator basis also makes the results directly applicable to phenomenological studies beyond the SM. They provide the necessary short-distance input for global analyses combining B -meson lifetimes and lifetime ratios with complementary flavour observables, such as mixing parameters and CP asymmetries, thereby allowing increasingly stringent constraints on BSM Wilson coefficients. The potential of combining lifetime observables with complementary flavour constraints has already been explored in Refs. [81, 86]. In particular, lifetime observables provide a complementary probe of possible new-physics effects in non-leptonic B decays and may help clarify the tensions observed in several colour-allowed decay modes between experiment and predictions based on QCD factorisation, see e.g. [92, 93].

Beyond a model-independent analysis, the matching coefficients derived here can be mapped onto specific ultraviolet scenarios, including models with additional charged gauge bosons, diquarks, or extended Higgs sectors. The results can also be adapted to scenarios with different colour structures after accounting for the corresponding colour factors, as illustrated, for example, in Ref. [94]. Further improvements, in particular higher-order QCD corrections to the chromomagnetic and Darwin coefficients and more precise determinations of the associated non-perturbative matrix elements, will further enhance the sensitivity of inclusive b -hadron observables to new physics.

Chapter 4

Phenomenology of B -Mesogenesis Models

“Collaborating with someone is like entering a relationship with that person.”

— Alexander Khodjamirian

The previous chapters were devoted to improving the HQE description of inclusive non-leptonic b -hadron decays within the SM and extending it to a model-independent effective Hamiltonian. We now shift our focus to the phenomenology of a specific BSM scenario, namely B -Mesogenesis.

The chapter begins with an introduction to the B -Mesogenesis framework, including its effective Hamiltonian and the existing constraints on the model parameter space. We then investigate the impact of the operators introduced by the model on the decay properties of B mesons, with particular emphasis on the lifetime ratios $\tau(B^+)/\tau(B_d)$ and $\tau(B_s)/\tau(B_d)$, as well as on the total B -meson lifetimes. The analysis is performed within the HQE, incorporating power-suppressed corrections up to order $1/m_b^3$, corresponding to dimension-six operators. In particular, we include both the dimension-six two-quark (Darwin) contribution and the dimension-six four-quark operators.

A distinctive feature of the B -Mesogenesis scenario is the presence of a massive fermion in the final state, whose mass can approach that of the b -quark. In this regime, the convergence of the OPE becomes nontrivial, as higher-order power corrections may become comparable to—or even exceed—the leading term. We therefore examine in detail the range of dark-fermion masses for which the OPE remains valid, and identify the point at which the HQE description breaks down. This analysis provides valuable insight into the theoretical limitations of the inclusive approach in scenarios involving heavy final states.

Finally, we perform an exclusive calculation of the decay of a B^+ meson into a proton and a dark-sector fermion, $B^+ \rightarrow p\psi$, using the method of LCSR. To this end, we compute the relevant $B \rightarrow p$ transition form factors based on B -meson distribution amplitudes and use them to predict the corresponding branching ratio (BR). By comparing the inclusive and exclusive predictions, we establish a quantitative connection between the two descriptions, enabling us to derive a lower bound on $\text{BR}(B^+ \rightarrow p\psi)$. This connection provides a robust phenomenological test of the B -Mesogenesis hypothesis and constrains its parameter space.

4.1 The Model

The B -Mesogenesis framework [30–33] offers an elegant and unified mechanism to explain both the observed baryon asymmetry of the Universe and the dark matter

abundance through processes involving B -meson dynamics in the early Universe.¹ In this scenario, the early Universe is assumed to be dominated by radiation and a weakly coupled scalar field Φ , which decays predominantly into $b\bar{b}$ pairs. The produced b -quarks hadronize into B and $\bar{B}$ mesons that subsequently decay not only through SM channels, but also via new, baryon-number-violating processes of the type $B \rightarrow \psi \mathcal{B}$, where $\mathcal{B}$ denotes a visible baryon and ψ is a dark-sector fermion carrying baryon number².

A baryon asymmetry arises from CP violation in neutral B_d - $\bar{B}_d$ and B_s - $\bar{B}_s$ meson mixing, which leads to unequal decay rates for B_q and $\bar{B}_q$ into final states containing $\mathcal{B}$ and ψ . As a result, the matter-antimatter imbalance of the visible sector is directly linked to the CP-violating parameters of the B -meson system.

The new decay mode $B \rightarrow \psi \mathcal{B}$ proceeds through the exchange of a heavy colour-triplet scalar field Y , which carries baryon number $-2/3$ and hypercharge $-1/3$ or $+2/3$. The most general renormalizable Lagrangian governing the interactions of Y with the SM quarks and the gauge-singlet fermion ψ can be written as:

$$\begin{aligned}\mathcal{L}_{-1/3} &= - \sum_{\alpha,\beta} y_{u\alpha d\beta} \epsilon_{ijk} Y^{*i} \bar{u}_{\alpha R}^j d_{\beta R}^{c,k} - \sum_{\gamma} y_{\psi d\gamma} Y_i \bar{\psi} d_{\gamma R}^{c,i} + \text{h.c.}, \\ \mathcal{L}_{+2/3} &= - \sum_{\alpha,\beta} y_{d\alpha d\beta} \epsilon_{ijk} Y^{*i} \bar{d}_{\alpha R}^j d_{\beta R}^{c,k} - \sum_{\gamma} y_{\psi u\gamma} Y_i \bar{\psi} u_{\gamma R}^{c,i} + \text{h.c.},\end{aligned}\tag{4.1}$$

where y denotes coupling constants, i, j, k are colour indices, α, β, γ label quark flavours, and the superscript c denotes charge conjugation. After integrating out the heavy colour-triplet Y at tree level, the resulting effective Hamiltonian mediating the B -meson decays can be written as

$$\begin{aligned}\mathcal{H}_{-1/3} &= \sum_{u,d} \left\{ \underbrace{\epsilon_{ijk} C_{\psi}^{(ud)} (\bar{u}_R^i b_R^{c,j}) (\bar{\psi} d_R^{c,k}) + \epsilon_{ijk} \tilde{C}_{\psi}^{(ud)} (\bar{u}_R^i d_R^{c,j}) (\bar{\psi} b_R^{c,k})}_{\text{(i) } \psi\text{-type operators}} \right. \\ &\quad \left. + \underbrace{\epsilon_{ijk} \epsilon_{imn} C_{uu}^{(d)} (\bar{u}_R^j b_R^{c,k}) (\bar{d}_R^{c,m} u_R^n) + C_{\psi\psi}^{(d)} (\bar{\psi} d_R^{c,i}) (\bar{b}_R^{c,i} \psi)}_{\text{(ii) hadronic and } \psi\psi \text{ operators}} \right\} + \text{h.c.},\end{aligned}\tag{4.2}$$

where $u \in \{u, c\}$, $d \in \{d, s\}$, and the Wilson coefficients, including the heavy scale suppression, are

$$C_{\psi}^{(ud)} = -\frac{y_{ub} y_{\psi d}}{M_Y^2}, \quad \tilde{C}_{\psi}^{(ud)} = -\frac{y_{ud} y_{\psi b}}{M_Y^2}, \quad C_{uu}^{(d)} = \frac{y_{ub} y_{ud}^*}{M_Y^2}, \quad C_{\psi\psi}^{(d)} = -\frac{y_{\psi d} y_{\psi b}^*}{M_Y^2}.\tag{4.3}$$

¹Several variations of this framework have been proposed in the literature, differing in their particle content—some even lacking a dark matter candidate [33]. In this work, however, we restrict ourselves to the specific realization presented in [31].

²The dark fermion ψ is not itself a dark matter candidate; it is required to decay rapidly into lighter, stable dark sector states (e.g., a scalar ϕ and a Majorana fermion ξ). These additional particles constitute the dark matter of the Universe but are irrelevant for the exotic B decays studied in this work and are therefore not discussed further.

For concreteness, we focus the detailed analysis on the operators proportional to $C_\psi^{(ud)}$ and $\tilde{C}_\psi^{(ud)}$. We follow the convention in Ref. [95] and refer to these operators as model (d) and model (b), respectively, and make the replacement $C_\psi^{(ud)} \rightarrow G_{(d)}$ and $\tilde{C}_\psi^{(ud)} \rightarrow G_{(b)}$. The calculations for the remaining operators have been fully performed and their results are provided in Section D.4, with references to them given at the relevant points in the main text.

A comprehensive analysis of the existing experimental and SM constraints on this model was performed in Ref. [31]. In the following, we briefly review those constraints that are most relevant for the lifetime analysis presented in this chapter.

1. Lower limit on inclusive exotic B decay: A key feature of the B -Mesogenesis mechanism is the direct connection between the observed baryon asymmetry and two collider observables: the semileptonic CP asymmetries A_{SL}^q in neutral B_q - $\bar{B}_q$ mixing, and the inclusive BR of the exotic decay $\text{BR}(B \rightarrow \psi X)$, where X denotes any allowed final state containing at least one visible baryon $\mathcal{B}$ and possibly additional light mesons. Given the experimental bound $|A_{\text{SL}}^{d,s}| \lesssim 10^{-3}$ at 95% CL [96], this imposes a lower bound on the inclusive exotic BR. To reproduce the observed baryon asymmetry, one requires

$$\boxed{\text{BR}(B \rightarrow \psi X) \gtrsim 10^{-4}.} \quad (4.4)$$

This bound is independent of the specific flavour structure of the effective operators in Eq. (4.2), and it holds for B^+ , B_d^0 , B_s^0 , and b -flavoured baryons alike. This constraint will be used later in this work to derive corresponding lower bounds on the exclusive decay channel $B^+ \rightarrow p^+ \psi$.

2. Effective couplings upper limits: The colour-triplet scalar mediator Y , which generates the effective operators in Eq. (4.2), can be produced at the LHC through its interactions with SM quarks. Searches for dijet resonances and jet plus missing transverse energy at ATLAS and CMS impose upper limits on the effective couplings. They will be used later in this work in the inclusive calculations of lifetimes and in deriving upper bounds on exclusive BR.

3. Single operator dominance: flavour-changing neutral current and mixing observables in the K , D , and B meson systems provide stringent constraints on the couplings of the colour-triplet scalar Y . In particular, box diagrams involving Y contribute to $\Delta F = 2$ processes — i.e., transitions that change a flavour quantum number (such as strangeness, charm, or bottomness) by two units — which govern neutral meson mixing. The absence of large deviations from the SM predictions forces the coupling in Eq. (4.1) to be highly hierarchical. Satisfying these flavour constraints while simultaneously achieving $\text{BR}(B \rightarrow \psi X) \gtrsim 10^{-4}$ requires that only one of the effective operators in Eq. (4.2) dominates, with all others strongly suppressed. Consequently, a single flavour combination—such as $(\psi b)(ud)$, $(\psi b)(us)$, $(\psi d)(ub)$, or $(\psi s)(ub)$ —is expected to be responsible for the entirety of the baryon asymmetry. This allows us to study the effect of each operator independently, without interference between different operator structures.

4. Dark sector ψ mass range: The mass of the dark fermion ψ is subject to several kinematic constraints. An upper bound follows from the requirement that the decay $B \rightarrow \psi X$ be kinematically allowed. The lightest visible baryon in

the final state is the proton ($m_p \approx 938$ MeV), giving $m_\psi < m_B - m_p \approx 4.34$ GeV. A lower bound is imposed by proton stability. If $m_\psi < m_p - m_e \approx 937.8$ MeV, the decay $p^+ \rightarrow \psi e^+$ would become kinematically accessible, leading to unobserved proton decay. This requires $m_\psi > m_p - m_e \approx 937.8$ MeV. Together, these constraints restrict the ψ mass to the window

$$\boxed{0.94 \text{ GeV} < m_\psi < 4.34 \text{ GeV}.} \quad (4.5)$$

All our results of the model analysis will be presented as functions of this dark sector mass range.

5. Experimental searches for $B^+ \rightarrow p + \text{invisible}$ decays: There have been dedicated efforts to search for baryonic B -meson decays with missing energy, specifically the channel $B^+ \rightarrow p + \text{invisible}$. The most recent analysis was performed by the Belle II collaboration using the Belle data sample [97], yielding upper limits on the branching fraction at the level of $\mathcal{O}(10^{-6})$. Earlier searches for the same decay mode were carried out by the BaBar collaboration [98], albeit with weaker sensitivity. In the following, these experimental upper bounds will be combined with our theoretical predictions for the lower limits on the exclusive branching fraction in order to constrain the allowed parameter space of the $B^+ \rightarrow p\psi$ decay channel.

4.2 Inclusive Calculations: HQE

The new decay modes introduced in the model necessarily modify B -meson observables. In particular, they contribute to the inclusive decay widths and, consequently, to the lifetime ratios. We study the effect of the new operators on these lifetime observables from an inclusive perspective, i.e., at the level of the quark transition $b \rightarrow \psi$. The analysis is performed within the framework of the HQE, outlined in Section 1.2. As presented in Eq. (1.14), the total decay rate of the B meson is expressed in terms of a leading dimension-three term representing the free quark decay, and $1/m_b$ power-suppressed terms of increasing dimension. We perform the calculations of the coefficients of the two-quark operators up to the dimension-six Darwin term, as well as the contribution of the dimension-six four-quark operators. Specifically, we compute the coefficients Γ_3 , Γ_5 , Γ_6 , and $\tilde{\Gamma}_6$, all at leading order in α_s .

We insert the operators of model (d) and model (b) from Eq. (4.2) into the time-ordered product in Eq. (1.13). The calculations follow the same procedures outlined in Chapters 2 and 3.

For the two-quark operator contributions, we contract all quark fields, leaving the bilinear $\bar{b}b$. The quark propagators are then expanded in accordance with Eq. (1.17) with the number of covariant derivatives included up to the desired accuracy. Since we perform the calculation up to dimension six, the quark propagator is expanded up to one covariant derivative acting on the gluon field strength tensor $G_{\mu\nu}$, yielding a total of three covariant derivatives sandwiched between the two b -quark fields — i.e., a dimension-six operator. Since the calculation procedures have been described in detail in Chapters 2 and 3, we do not repeat them here. However, the new B -Mesogenesis operators contain a particular structure involving charge-conjugated fields and nontrivial colour contractions, to which we pay special attention.

In performing the Wick contraction to obtain the two-quark contribution, one encounters expressions of the form

$$: \epsilon_{ijk} \epsilon_{mnr} \left(\bar{b}^{m,c} \overline{u^n}^i(x) b^{j,c}(x) \right) \left(\overline{\psi(x) d^{r,c}(x) d^{k,c} \psi} \right) : \quad (4.6)$$

Using the definition of the charge-conjugated fields and the properties in Eqs. (A.16) and (A.17), this can be reduced to

$$\begin{aligned} & \epsilon_{ijk} \epsilon_{mnr} \left(\bar{b}^j(x) (S_{ni}^{(u)}(0, x))^T b^m \right) \text{Tr} \left(C P_L^T (S_{rk}^{(d)}(0, x))^T P_R^T C^{-1} S^{(\psi)}(0, x) \right) \\ &= \epsilon_{ijk} \epsilon_{mnr} \left(\bar{b}^j(x) (S_{ni}^{(u)}(0, x))^T b^m \right) \text{Tr} \left(P_L S_{rk}^{(d)}(0, x) S^{(\psi)}(0, x) \right), \end{aligned} \quad (4.7)$$

where C is the charge conjugation operator and the following relation holds:

$$\overline{q^T(x)} q^T(0) = -(S^{(q)}(0, x))^T. \quad (4.8)$$

Since the power-suppressed contributions arise from the emission of a soft gluon from the internal quark propagators, the associated colour structure becomes decisive. In the present case, the gluon can be emitted either from the u - or from the d -quark line. For emission from the u -quark propagator one has $S_{ni}^{(u)} \rightarrow t_{ni}^a$ and $S_{rk}^{(d)} \rightarrow \delta_{rk}$, leading to the colour factor

$$\epsilon_{ijk} \epsilon_{mnr} \delta_{kr} t_{ni}^a = \epsilon_{ijk} \epsilon_{mnk} t_{ni}^a = t_{jm}^a. \quad (4.9)$$

An analogous contraction for emission from the d -quark line, $S_{rk}^{(d)} \rightarrow t_{rk}^a$ and $S_{ni}^{(u)} \rightarrow \delta_{ni}$, yields the same result. Thus, soft-gluon emission from both the u - and d -quark propagators contribute and must be included consistently in the power-suppressed analysis.

We now turn to the contribution of the dimension-six four-quark operators, corresponding to the coefficients $\tilde{\Gamma}_6$ in Eq. (1.14). In contrast to the two-quark operator case discussed above, the structure of the Wick contractions is different. Here, only one quark line is contracted in addition to the dark-sector fermion ψ , while the remaining light quark fields remain uncontracted and give rise to local four-quark operators in the effective theory.

For model (d), the relevant time-ordered product (suppressing colour indices) reads

$$\begin{aligned} & T^{(d)} \left\{ \bar{u}_R(x) b_R^c(x) \bar{\psi}(x) d_R^c(x), \bar{b}_R^c u_R \bar{d}_R^c \psi \right\} \\ &= : \bar{u}_R(x) b_R^c(x) \overline{\psi(x) d_R^c(x) \bar{b}_R^c u_R \bar{d}_R^c \psi} : \\ &+ : \overline{\bar{u}_R(x) b_R^c(x) \psi(x) d_R^c(x) \bar{b}_R^c u_R \bar{d}_R^c \psi} : \end{aligned} \quad (4.10)$$

The contractions for model (b) proceed analogously. The corresponding topologies are shown in Figure 4.1. As an illustrative example, we discuss in detail the evaluation of the first contraction in Eq. (4.10). Inserting the effective operator of model (d) in

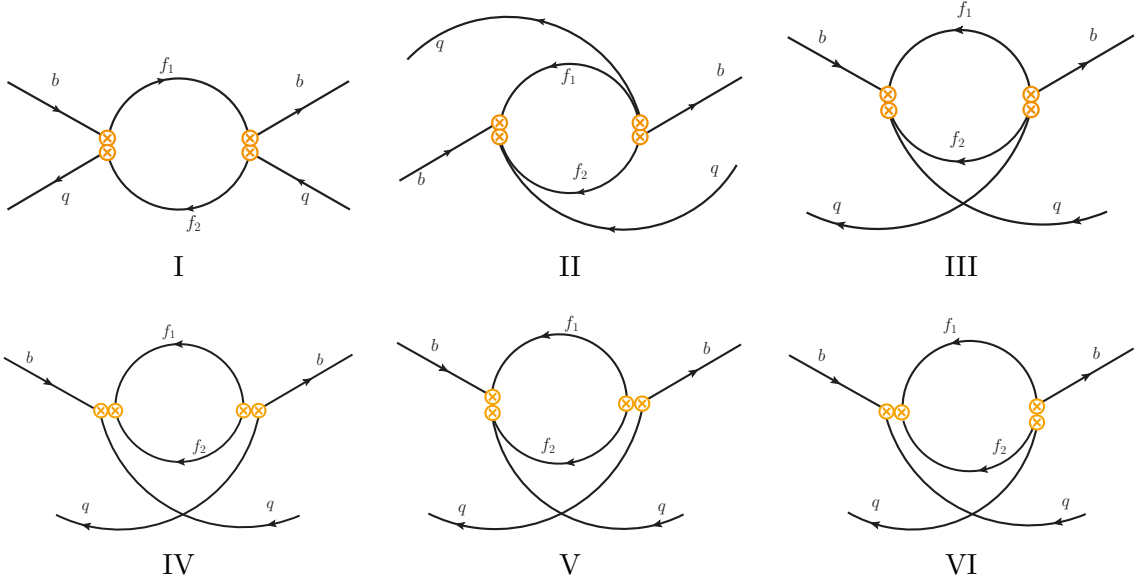


Figure 4.1: Complete set of four-quark topologies (see Eq. (4.10)) induced by the Mesogenesis operators in Eq. (4.1). The internal fermions f_1 and f_2 are generic and may correspond to any of the particles appearing in the Mesogenesis operators, namely ψ , u , d , s , or c .

Eq. (4.2) into Eq. (1.13), the dimension-six four-quark operators contribution to the total decay rate of topology-IV reads

$$\Gamma_{(6,d),IV}^{(4q)}(B) = \frac{-|G_{(d)}|^2}{2m_B} \text{Im} \int d^4x \langle B | i \mathcal{T}_{(d)}^{IV}(x) | B \rangle + (x \leftrightarrow 0), \quad (4.11)$$

where we define for convenience

$$\mathcal{T}_{(d)}^{IV}(x) = \epsilon_{ijk} \epsilon_{mnr} \text{Tr} \left\{ \bar{u}_R^i(x) b_R^{c,j}(x) \bar{\psi}(x) d_R^{c,k}(x) \bar{b}_R^{c,m} u_R^n \bar{d}_R^{c,r} \psi \right\}. \quad (4.12)$$

Performing the Wick contractions and using

$$\overline{d_R^{c,k}(x) \bar{d}_R^{c,r}(0)} = C P_L^T (i S_{rk}^{(d)}(0, x))^T P_R^T C^{-1}, \quad (4.13)$$

we obtain

$$\mathcal{T}_{(d)}^{IV}(x) = -\epsilon_{ijk} \epsilon_{mnr} \bar{u}_R^i(x) b_R^{c,j}(x) \bar{b}_R^{c,m} u_R^n \text{Tr} \left[C P_L^T (i S_{rk}^{(d)}(0, x))^T P_R^T C^{-1} i S^{(\psi)}(0, x) \right]. \quad (4.14)$$

The overall minus sign originates from the anticommuting nature of the fermion fields in the Wick contraction; in diagrammatic language, this is equivalently represented by the standard factor (-1) associated with a closed fermion loop. Substituting the tree-level propagator in Eq. (D.1) (see also Eq. (1.16)), we arrive at

$$\mathcal{T}_{(d)}^{IV}(x) = - \int_{l_1 l_2} \frac{l_1^\alpha l_2^\beta e^{ix(l_1+l_2)}}{[l_1^2 - m_\psi^2][l_2^2 - m_d^2]} \epsilon_{ijk} \epsilon_{mnk} \bar{u}_R^i(x) b_R^{c,j}(x) \bar{b}_R^{c,m} u_R^n \text{Tr} \left[P_L \gamma_\alpha \gamma_\beta \right], \quad (4.15)$$

where we define $\int_{l_1 l_2} \equiv \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4}$, l_1 and l_2 denote the four-momenta flowing in the ψ - and d -quark propagators, respectively. The Dirac bilinear in front of the trace in Eq. (4.15) still involves charge-conjugated fields. To rewrite it in terms of ordinary quark bilinears, we apply a Fierz rearrangement for chiral spinors (see, e.g., Ref. [99]). In four dimensions one finds

$$\bar{u}_R^i(x) b_R^{c,j}(x) \bar{b}_R^{c,m} u_R^n = \frac{1}{2} [\bar{b}^j(x) \gamma_\mu P_R b^m] [\bar{u}^i(x) \gamma^\mu P_R u^n]. \quad (4.16)$$

In order to match onto the forward matrix element (see Eqs. (D.3) and (D.4)), we perform a second Fierz rearrangement,

$$[\bar{b}^j(x) \gamma_\mu P_R b^m] [\bar{u}^i(x) \gamma^\mu P_R u^n] = [\bar{b}^j(x) \gamma_\mu P_R u^n] [\bar{u}^i(x) \gamma^\mu P_R b^m]. \quad (4.17)$$

Since we evaluate the forward matrix element at parton level between free quark states, we insert momentum eigenstates and use the plane-wave expansion of the fields,

$$\bar{b}(x) \rightarrow \bar{b}(0) e^{ip_b x}, \quad \bar{u}(x) \rightarrow u(0) e^{iqx},$$

where p_b and q denote the momenta of the corresponding quark lines. Defining $p = p_b + q$ and using Eqs. (4.16) and (4.17), and performing the trace, Eq. (4.15) becomes

$$\mathcal{T}_{(d)}^{IV}(x) = -g_{\alpha\beta} \int_{l_1 l_2} \frac{l_1^\alpha l_2^\beta e^{ix(l_1+l_2+p)}}{[l_1^2 - m_\psi^2][l_2^2 - m_d^2]} \epsilon_{ijk} \epsilon_{mnk} [\bar{b}^j \gamma_\mu P_R u^n] [\bar{u}^i \gamma^\mu P_R b^m], \quad (4.18)$$

The colour structure can now be simplified using the identity

$$\epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}. \quad (4.19)$$

In addition, we make use of the standard relation for the $SU(3)$ generators t^a in the fundamental representation,

$$t_{ij}^a t_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right), \quad (4.20)$$

with $N_c = 3$. Applying these identities to Eq. (4.18) allows the four-quark structure to be decomposed into colour-singlet and colour-octet contributions

$$\begin{aligned} \epsilon_{ijk} \epsilon_{mnk} [\bar{b}^j \gamma_\mu P_R u^n] [\bar{u}^i \gamma^\mu P_R b^m] &= \left(\frac{N_c - 1}{N_c} \right) [\bar{b}^i \gamma_\mu P_R u^i] [\bar{u}^j \gamma^\mu P_R b^j] \\ &\quad - 2 [\bar{u}^i \gamma^\mu P_R t_{ij}^a b^j] [\bar{u}^m \gamma_\mu P_R t_{mn}^a b^n]. \end{aligned} \quad (4.21)$$

Substituting Eq. (4.21) into Eq. (4.18) and using the four-quark operators definitions in Eq. (D.3)

$$\mathcal{T}_{(d)}^{IV}(x) = \frac{-g_{\alpha\beta}}{4} \int_{l_1 l_2} \frac{l_1^\alpha l_2^\beta e^{ix(l_1+l_2+p)}}{[l_1^2 - m_\psi^2][l_2^2 - m_d^2]} \left[\left(\frac{N_c - 1}{N_c} \right) \tilde{\mathcal{O}}_1^{(u)} - 2 \tilde{\mathcal{O}}_3^{(u)} \right]. \quad (4.22)$$

The operators $\tilde{\mathcal{O}}_1^{(u)}$ and $\tilde{\mathcal{O}}_3^{(u)}$ are the four-quark operators defined in Eq. (D.3). Substituting Eq. (4.22) back into Eq. (4.11), we then perform the x -integration,

which yields $(2\pi)^4 \delta^{(4)}(l_1 + l_2 + p)$, and use it to integrate over l_2 . Including the symmetry factor 2 from $(x \leftrightarrow 0)$ and relabeling $l_1 \rightarrow l$, we arrive at

$$\begin{aligned} \Gamma_{(6,d),IV}^{(4q)}(B) &= \frac{|G_{(d)}|^2}{4m_B} g_{\alpha\beta} \text{Im} \left(i \int_l \frac{l^\alpha (-l-p)^\beta}{[l^2 - m_\psi^2][(-l-p)^2 - m_d^2]} \right) \\ &\times \langle B | \left[\left(\frac{N_c - 1}{N_c} \right) \tilde{\mathcal{O}}_1^{(u)} - 2 \tilde{\mathcal{O}}_3^{(u)} \right] | B \rangle. \end{aligned} \quad (4.23)$$

To match the convention of [Eq. \(D.2\)](#), we relabel the loop momentum by $l \rightarrow -l$. Evaluating the imaginary part of the integration and contracting the Dirac indices, the dimension-six four-quark operator for topology-IV for model (d) reads

$$\Gamma_{(6,d),IV}^{(4q)}(B) = \frac{-|G_{(d)}|^2}{128\pi m_B} \left(\frac{m_b^2 - m_\psi^2}{m_b} \right)^2 \langle B | \left[\left(\frac{N_c - 1}{N_c} \right) \tilde{\mathcal{O}}_1^{(u)} - 2 \tilde{\mathcal{O}}_3^{(u)} \right] | B \rangle. \quad (4.24)$$

Having outlined the calculation strategy for both the two-quark and the dimension-six four-quark contributions in the context of the Mesogenesis operators for representative examples, the remaining operator structures proceed in complete analogy. The steps — Wick contraction, treatment of charge-conjugated fields, Fierz rearrangements, colour decomposition, and evaluation of the loop integrals — follow the same pattern and introduce no qualitatively new features. We are therefore now in a position to present the general results for the inclusive decay width induced by the Mesogenesis effective operators.

Inclusive Results from the HQE

For B -meson decays mediated by the B -Mesogenesis operators in [Eq. \(4.2\)](#), the inclusive decay width corresponding to a given operator structure i can be expressed within the HQE as the sum of the two-quark and dimension-six four-quark contributions:

$$\Gamma_{\text{meso}}^{(i)} = \Gamma_{\text{meso}, 2q}^{(i)} + \Gamma_{\text{meso}, 4q}^{(i)} + \mathcal{O}\left(\frac{1}{m_b^4}\right). \quad (4.25)$$

The two-quark part encodes the free partonic decay and the power-suppressed corrections from dimension-five and dimension-six two-quark operators and can be expressed in the compact form

$$\Gamma_{\text{meso}, 2q}^{(i)} = \frac{|C^{(i)}|^2 m_b^5}{3072 \pi^3} \left[f_0^{(i)}(\rho_f) \left(1 - \frac{\mu_\pi^2}{2m_b^2} \right) + f_G^{(i)}(\rho_f) \frac{\mu_G^2}{m_b^2} + f_D^{(i)}(\rho_f) \frac{\rho_D^3}{m_b^3} \right], \quad (4.26)$$

with $\rho_f \equiv (m_f/m_b)^2$ denoting the squared mass ratio of the massive final-state particle in the underlying transition $b \rightarrow f$. The functions $f_0^{(i)}(\rho_f)$, $f_G^{(i)}(\rho_f)$, and $f_D^{(i)}(\rho_f)$ are dimensionless short-distance coefficient functions that depend on the Dirac and colour structure of the corresponding B -Mesogenesis operator³. As an

³The spin-orbit parameter ρ_{LS}^3 does not contribute in the present case, as the associated operator yields a vanishing forward matrix element for the Dirac structures under consideration.

example, we display the results for models (d) and (b), which correspond to the coefficients $C_\psi^{(ud)}$ and $\tilde{C}_\psi^{(ud)}$, respectively:

$$f_0^{(\psi ud)}(\rho_\psi) = 1 - 8\rho_\psi + 8\rho_\psi^3 - \rho_\psi^4 - 12\rho_\psi^2 \log \rho_\psi, \quad (4.27)$$

$$f_G^{(\psi ud)}(\rho_\psi) = \frac{5}{2} - 8\rho_\psi + 8\rho_\psi^3 - \frac{5}{2}\rho_\psi^4 - 6\rho_\psi^2 \log \rho_\psi, \quad (4.28)$$

$$f_D^{(\psi ud)}(\rho_\psi) = \frac{2}{3} \left(22 - 31\rho_\psi + 3\rho_\psi^2 + 11\rho_\psi^3 - 5\rho_\psi^4 \right. \\ \left. - 12(1 - \rho_\psi)^2(3 + \rho_\psi) \log(1 - \rho_\psi) + 6\rho_\psi^2(1 + \rho_\psi) \log \rho_\psi \right), \quad (4.29)$$

$$f_0^{(\tilde{\psi} ud)}(\rho_\psi) = 1 - 8\rho_\psi + 8\rho_\psi^3 - \rho_\psi^4 - 12\rho_\psi^2 \log \rho_\psi, \quad (4.30)$$

$$f_G^{(\tilde{\psi} ud)}(\rho_\psi) = -\frac{1}{2} \left(3 - 8\rho_\psi + 24\rho_\psi^2 - 24\rho_\psi^3 + 5\rho_\psi^4 + 12\rho_\psi^2 \log \rho_\psi \right), \quad (4.31)$$

$$f_D^{(\tilde{\psi} ud)}(\rho_\psi) = \frac{2}{3} (1 - \rho_\psi) \left(9 + 11\rho_\psi - 25\rho_\psi^2 + 5\rho_\psi^3 \right. \\ \left. - 24(1 - \rho_\psi^2) \log(1 - \rho_\psi) - 12\rho_\psi^2 \log \rho_\psi \right). \quad (4.32)$$

The results for the remaining operators are given in [Section D.4.1](#).

For a fixed operator, the dimension-six four-quark contribution depends on the external meson species. Since the two possible Wick contractions correspond to different spectator flavours, they contribute to different B mesons and are not summed over. Explicitly, we write

$$\Gamma_{\text{meso}, 4q}^{(i)}(B^+) = \frac{|C^{(i)}|^2 m_b^2}{192\pi} \sum_{a=1}^4 \tilde{\Gamma}_{X,a}^{(i,u)} \frac{\langle B^+ | \tilde{\mathcal{O}}_a^u | B^+ \rangle}{2m_B}, \quad (4.33)$$

$$\Gamma_{\text{meso}, 4q}^{(i)}(B_q) = \frac{|C^{(i)}|^2 m_b^2}{192\pi} \sum_{a=1}^4 \tilde{\Gamma}_{X,a}^{(i,q)} \frac{\langle B_q | \tilde{\mathcal{O}}_a^q | B_q \rangle}{2m_B}, \quad q = d, s. \quad (4.34)$$

Here X labels the topology arising from the operator structure (see [Figure 4.1](#)). The explicit expressions for $\tilde{\Gamma}_{X,a}^{(i,u)}$ and $\tilde{\Gamma}_{X,q}^{(i,q)}$ are given in [Section D.4.1](#).

We stress that the analytical results presented in this section have been subjected to several non-trivial cross-checks. The four-quark contributions were independently reproduced in a separate calculation, providing an important validation of the corresponding Wilson coefficients. Furthermore, for the two-quark contributions, we explicitly verified the cancellation of the IR divergences against the corresponding four-quark contributions through the operator-mixing procedure described in [Chapters 2 and 3](#). After accounting for the appropriate colour factors and Dirac structures, we also reproduce the known SM results [\[63, 100\]](#), with ρ_ψ in the present notation corresponding to ρ_c in the SM. A detailed comparison between the SM and the present operator basis is presented in [Section D.2](#).

4.2.1 Lifetimes and Lifetime Ratios

Using the analytic results derived in the previous section, we now perform a numerical analysis of the B -meson lifetimes and lifetime ratios within the framework of the HQE.

Total Lifetimes

In general, the analysis of total lifetimes is not expected to provide stringent constraints on the couplings of the B -Mesogenesis model. The SM predictions for total lifetimes are affected by sizeable theoretical uncertainties, predominantly from non-perturbative inputs, see e.g. [42, 101, 102]. Moreover, collider searches already restrict the relevant BSM couplings to comparatively small values. Consequently, the primary purpose of the total lifetime analysis is not to derive competitive bounds, but rather to investigate the impact of the subleading power-suppressed contributions induced by the new operators. In particular, we study the hierarchy of the OPE, identifying the range of m_ψ where subleading terms become relevant and determining which subleading term provide the dominant contributions. We emphasize that, in the following analysis of the total decay rate, we focus exclusively on the new-physics (NP) contributions induced by the Mesogenesis operators, omitting the SM part of the width. This enables us to isolate and assess the relative importance of the subleading HQE terms within the BSM framework.⁴

The total B -meson decay rate is given in Eq. (1.14). In the previous section, we computed the two-quark contributions up to the dimension-six Darwin term, together with the four-quark operator contributions. These enter the HQE coefficients Γ_3 , Γ_5 , Γ_6 , and $\tilde{\Gamma}_6$.

In the following, we analyse, for each operator in the effective Hamiltonian Eq. (4.2), the individual contributions of the HQE terms in Eq. (1.14) to the total decay width of the B^+ meson. We concentrate on the B^+ system since it exhibits distinctive topological structures and, as will be shown below, enables us to derive a relevant experimental lower bound for the exclusive decay $B^+ \rightarrow p\psi$ (see Section 4.3). In general, the HQE contributions from two-quark operators are spectator independent and therefore nearly identical for all B mesons (B^+ , B^0 , B_s^0), up to $SU(3)_F$ -breaking effects in the non-perturbative matrix elements. In contrast, the four-quark operator contributions are suppressed for the neutral mesons B^0 and B_s^0 . Moreover, certain operators in Eq. (4.2)—particularly those involving two additional massive fermions besides the b quark—do not contribute to the B^+ decay width. For completeness, the corresponding contributions to the B^0 width are derived in Section D.4.

As in the previous section, we concentrate on models (d) and (b) for definiteness. For each case, we present in Figure 4.2 the individual contributions to $\Gamma(B^+)$ from the leading dimension-three term Γ_3 , the dimension-five correction Γ_5 , the Darwin term Γ_6 , and the dimension-six four-quark contributions $\tilde{\Gamma}_6$, as functions of $\rho_f = (m_f/m_b)^2$. To quantify the relative size of subleading effects, we introduce $R_d \equiv |\Gamma_d/\Gamma_3|$ where Γ_d

⁴For coupling values consistent with current ATLAS and CMS bounds [31], the NP contribution amounts to at most $\sim 3\%$ of the SM total width and is therefore negligible compared to the present experimental and theoretical uncertainties.

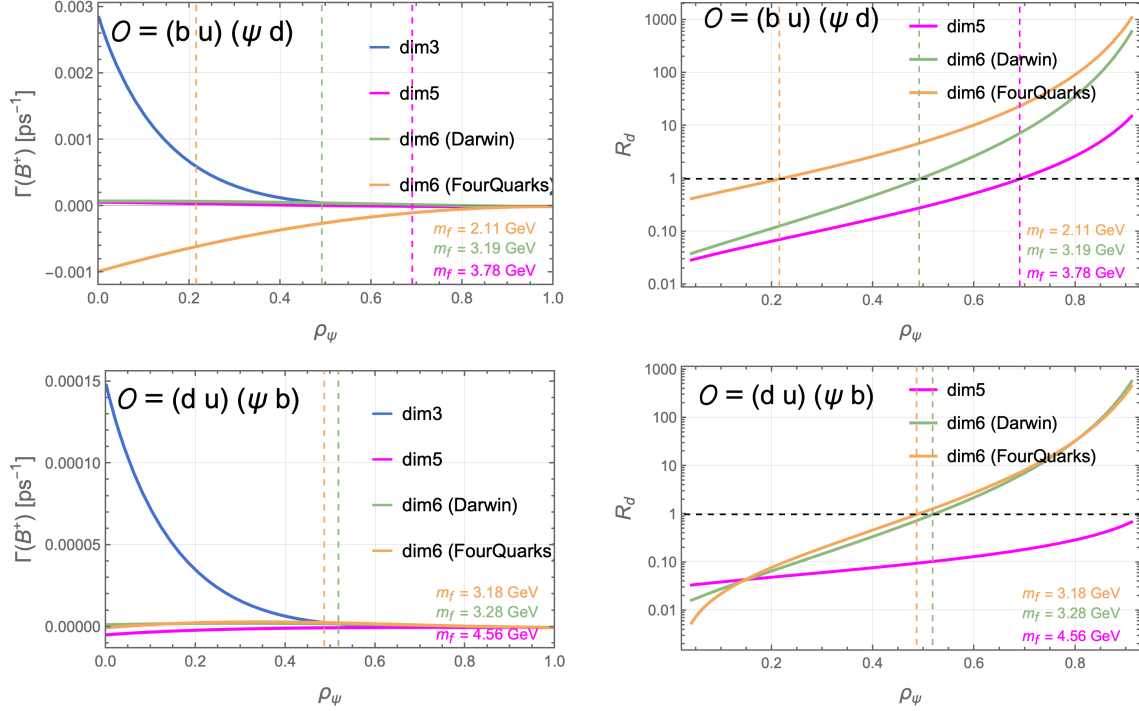


Figure 4.2: Total decay width $\Gamma(B^+)$ induced solely by the Mesogenesis operators as a function of $\rho_f = (m_f/m_b)^2$. The left column shows the individual HQE contributions, while the right column displays the ratio $R_d = |\Gamma_d/\Gamma_3|$, where Γ_d denotes the corresponding subleading term. The vertical lines indicate the values of ρ_f at which the subleading contribution exceeds the leading dimension-three term Γ_3 ; the associated m_f values are also shown.

denotes one of the subleading contributions listed above. We also show in Figure 4.3 the total decay width, obtained by summing all contributions, as a function of ρ_f . For convenience, the numerical inputs used in this analysis are collected in Section D.1. The analysis leads to the following key observations:

- **Hierarchy of contributions:** In general, we find $\tilde{\Gamma}_6 > \Gamma_6 > \Gamma_5$, i.e. the dimension-six four-quark contributions dominate over the Darwin term, which in turn exceeds the dimension-five corrections. This hierarchy closely mirrors the pattern observed in the SM, where the dominant dimension-six four-quark effects originate from PI topologies [63, 102]. The dominance of $\tilde{\Gamma}_6$ can be traced back to the phase-space and loop enhancement of four-quark spectator contributions (the familiar $16\pi^2$ factor, cf. Eq. (1.14)), as well as to the specific colour and bag-parameter combinations entering the matrix elements (see Section D.2). The relevant dimension-six four-quark topologies for the two representative cases are illustrated in Figure 4.4.
- **Operator-dependent behaviour:** For the operator $\mathcal{O} = (b u^c)(\psi d^c)$, the dimension-six four-quark contribution exceeds the leading dimension-three term over a significant portion of the ρ_ψ range. This behaviour is characteristic of operators of the form $\mathcal{O} = (b q_1^c)(q_2 q_3^c)$, where q_1 denotes the spectator quark and q_2, q_3 are arbitrary fermions. Such structures generate a $\overline{\text{PI}}$ topology (see

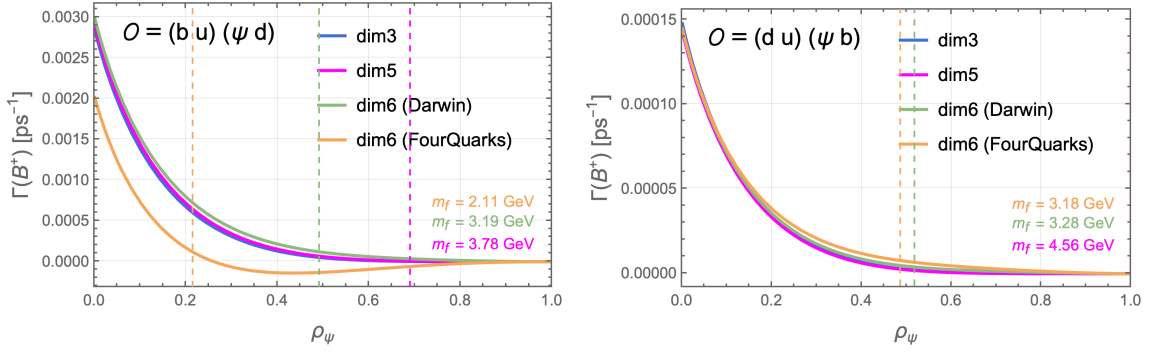


Figure 4.3: Total decay width of $\Gamma(B^+)$ arising purely from the Mesogenesis operators as a function of $\rho_f = (m_f/m_b)^2$. The figure displays the cumulative sum of all terms up to the indicated dimension. The vertical lines indicate the value of ρ_f , where the contribution exceeds the leading dimension-three Γ_3 term (the corresponding m_f values are also indicated).

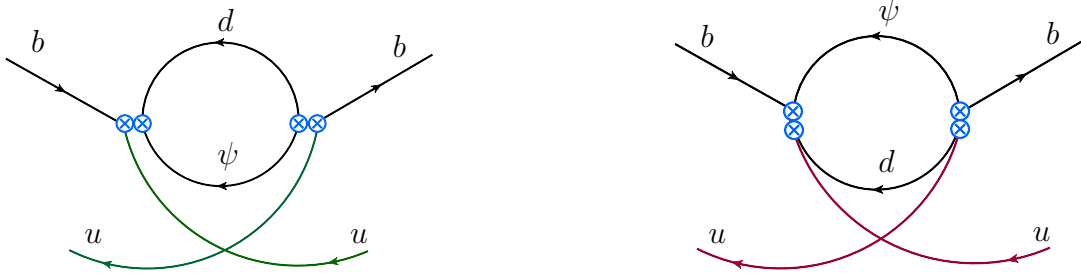


Figure 4.4: Topologies of the four-quark contributions to the total decay width $\Gamma(B^+)$ for the Mesogenesis models: model (d) (left, $\overline{\text{PI}}$) and model (b) (right, PI). The respective operators are $\mathcal{O} = (b u)(\psi d)$ and $\mathcal{O} = (d u)(\psi b)$.

Figure 4.4), which can induce a sizeable negative contribution to the decay width. A similar pattern appears in $\Gamma(B_d)$ for operators arising from $\mathcal{L}_{2/3}$, for instance $\mathcal{O} = (b d^c)(\psi u^c)$, where the spectator quark couples directly to the b quark, again leading to the $\overline{\text{PI}}$ topology.

- **Comparison to the SM:** In the SM, operators such as $\mathcal{O} = (u b)(d u)$ also generate PI topologies. However, in that case the spectator quark couples to a light quark rather than directly to the b quark. For operators of the form $\mathcal{O} = (u b)(d c)$, the b quark couples to the spectator, but the resulting contraction does not produce a PI topology. In contrast, the Mesogenesis operators involve charge-conjugated fields, leading to a modified $\overline{\text{PI}}$ topology that circumvents certain colour and contraction suppressions present in the SM. This structural difference explains the enhanced behaviour observed in the dimension-six contributions. A detailed operator-level comparison is provided in [Section D.2](#).

We therefore conclude that the hierarchy of HQE contributions can be strongly operator dependent. In particular, models involving non-SM operator structures

should be examined carefully for such effects before drawing phenomenological conclusions.

It is worth emphasizing, however, that the present analysis is performed at leading order (LO). Experience from the SM indicates that next-to-leading order (NLO) QCD corrections and higher-dimensional contributions can alter the relative size of individual terms (see, e.g., Ref. [102]). The hierarchy observed here should thus be interpreted as an LO indication, subject to quantitative refinement in more complete analyses.

Lifetime Ratios

Experimentally, the lifetime ratios of B mesons are measured with remarkable precision (see e.g. the HFLAV review [103]), while the corresponding HQE predictions have reached a comparable level of theoretical control, see e.g. Ref. [37, 102]:

$$\frac{\tau(B^+)_{\text{Exp.}}}{\tau(B_d^0)} = 1.076(4), \quad \frac{\tau(B_s^0)_{\text{Exp.}}}{\tau(B_d^0)} = 1.0021(34), \quad (4.35)$$

$$\frac{\tau(B^+)_{\text{HQE}}}{\tau(B_d^0)} = 1.072 \pm 0.024, \quad \frac{\tau(B_s^0)_{\text{HQE}}}{\tau(B_d^0)} = 1.013(7). \quad (4.36)$$

We observe an excellent agreement between experiment and HQE predictions. In contrast to the total lifetimes, the theoretical uncertainties in the ratios are significantly reduced, since dominant hadronic inputs largely cancel. The remaining difference between experimental and theoretical precision amounts only to a factor of a few, making lifetime ratios particularly sensitive probes of subleading HQE contributions and potential new-physics effects.

In the HQE framework, the lifetime ratio can be expressed in terms of the difference of total decay widths. In particular,

$$\frac{\tau(B_s^0)}{\tau(B_d^0)} = \frac{\Gamma(B_d^0)}{\Gamma(B_s^0)} = 1 + \tau(B_s^0)_{\text{Exp.}} [\Gamma(B_d^0) - \Gamma(B_s^0)]^{\text{HQE}}. \quad (4.37)$$

Expanding the width difference in powers of $1/m_b$ (see e.g. Eq. (1.14)), one obtains

$$\begin{aligned} \frac{\tau(B_s^0)}{\tau(B_d^0)} = 1 + \tau(B_s^0)_{\text{Exp.}} & \left[\Gamma_5 \frac{\langle O_5 \rangle_{B_d^0} - \langle O_5 \rangle_{B_s^0}}{m_b^2} + \Gamma_6 \frac{\langle O_6 \rangle_{B_d^0} - \langle O_6 \rangle_{B_s^0}}{m_b^3} \right. \\ & \left. + 16\pi^2 \left(\frac{(\tilde{\Gamma}_6 \langle \tilde{O}_6 \rangle)_{B_d^0} - (\tilde{\Gamma}_6 \langle \tilde{O}_6 \rangle)_{B_s^0}}{m_b^3} + \frac{(\tilde{\Gamma}_7 \langle \tilde{O}_7 \rangle)_{B_d^0} - (\tilde{\Gamma}_7 \langle \tilde{O}_7 \rangle)_{B_s^0}}{m_b^4} + \dots \right) \right]. \end{aligned} \quad (4.38)$$

The leading dimension-three contribution, Γ_3 , cancels in the ratio $\tau(B_s^0)/\tau(B_d^0)$, since it is universal for both mesons.⁵ As a consequence, the ratio is governed by differences of subleading terms in the HQE, starting at dimension five.

⁵This cancellation removes the strong m_b^5 dependence of the total decay width, substantially reducing the theoretical uncertainty.

For the ratio $\tau(B^+)/\tau(B_d^0)$, an especially strong cancellation occurs. In the isospin limit ($m_u = m_d$ and identical non-perturbative matrix elements), all two-quark contributions cancel in the ratio. As a result, the lifetime difference between charged and neutral B mesons is predominantly driven by spectator-dependent effects arising from dimension-six four-quark operators. Unlike the two-quark terms, both the short-distance Wilson coefficients $\bar{\Gamma}_{d=6,7,\dots}$ and the associated hadronic matrix elements depend explicitly on the flavour content of the external hadron. Consequently, they differ for B_d^0 , B_s^0 , and B^+ mesons and can generate sizeable lifetime splittings.

For the ratio $\tau(B_s^0)/\tau(B_d^0)$, however, spectator effects are numerically small [101]. This suppression originates from the approximate $SU(3)_F$ symmetry between the d - and s -spectator quarks, which reduces differences in the relevant matrix elements. In contrast, the ratio $\tau(B^+)/\tau(B_d^0)$ exhibits a sizeable effect of order $\mathcal{O}(10\%)$ [101]. Here, PI between the spectator quark and the decay products of the b quark produces a substantial dimension-six contribution. This mechanism distinguishes charged from neutral B mesons and dominates their lifetime difference.

For comparison, analogous spectator effects are even more pronounced in the charm sector, where they exceed 100% in the ratio $\tau(D^+)/\tau(D^0)$ [104], reflecting the slower convergence of the HQE for charm hadrons. For more details on the charm sector, see e.g. the review [105].

The primary goal of the lifetime-ratio analysis is to determine whether the measured B -meson lifetimes can provide additional constraints on the couplings of the B -Mesogenesis model. Existing bounds on these couplings have already been obtained from ATLAS and CMS searches for colour-triplet scalars [31]. Throughout this analysis, we therefore adopt the corresponding upper limits on the couplings. Using these benchmark values, we compute the lifetime ratios as a function of the dark-fermion mass m_ψ . The corresponding results are shown in Section D.4.3. In each case, a single coupling is assumed to be non-zero, while all remaining couplings are set to zero. We find that, within the current theoretical and experimental uncertainties, the predicted lifetime ratios remain consistent with the experimental measurements over the entire range of m_ψ . Consequently, the lifetime ratios $\tau(B_s^0)/\tau(B_d^0)$ and $\tau(B^+)/\tau(B_d)$ do not presently provide constraints on the couplings beyond those already obtained from direct collider searches.

It is nevertheless instructive to compare the implications of constraining the Mesogenesis couplings using lifetime ratios with those obtained from direct collider searches and flavour-mixing observables. To this end, we determine the corresponding predictions for the inclusive B -meson branching ratios using the couplings constrained by each of these approaches. Besides the bounds from collider searches, Ref. [31] also derives constraints from flavour physics. These are obtained by evaluating the contributions of the Mesogenesis operators to the neutral-meson mixing observables ϵ_K and $\Delta M_{B,K}$ and comparing the resulting predictions with the corresponding experimental measurements and SM expectations.

To compute the inclusive BRs, we evaluate the leading HQE contribution, Γ_3 , and normalize it using the experimental lifetime of the B^+ meson. In doing so, we implicitly assume that the SM prediction for the total decay width agrees with the inverse of the measured lifetime. This assumption is well justified, as the total

Operator	Affects B^+ ?	Affects B_d ?	Topology	BR ($\tau(B^+)/\tau(B_d)$)
$(\bar{b}d^c)(\bar{\psi}u^c)$	Yes	Yes	III (B^+), IV (B_d)	Figure D.15 (a)
$(\bar{b}d^c)(\bar{\psi}c^c)$	No	Yes	IV	Figure D.15 (b)
$(\bar{b}d^c)(\bar{d}s^c)$	No	Yes	{I, IV}	Figure D.15 (c)
$(\bar{b}s^c)(\bar{d}s^c)$	No	Yes	I	Figure D.15 (d)
$(\bar{b}s^c)(\bar{\psi}u^c)$	Yes	No	III	Figure D.15 (e)

Table 4.1: $\mathcal{L}_{2/3}$ operator contributions to B^+ and B_d meson decay rates from dimension-6 four-quark operators, showing topologies and inclusive BRs using couplings constrained by $\tau(B^+)/\tau(B_d)$. Rows highlighted in red correspond to the least-constrained operators.

B -meson decay widths are currently predicted within the SM with a precision of about 5% [102] and are in good agreement with experiment. Consequently, any additional decay channel contributing more than approximately 5% to the total width would be incompatible with the measured lifetime. Such parameter points are therefore excluded and are not shown in the branching-ratio plots beyond this limit.

This procedure allows us to compare the phenomenological implications of the different constraints on the Mesogenesis couplings. In particular, we determine the inclusive branching ratios obtained when the couplings are constrained by lifetime ratios, direct collider searches, and flavour-mixing observables, respectively. The corresponding allowed parameter regions and branching-ratio predictions are presented in Figure D.14–D.19 for the $\mathcal{H}_{2/3}$ and $\mathcal{H}_{-1/3}$ scenarios.

Figures D.14, D.16 and D.17, show the regions of the $\{G_X^{f_1 f_2}, m_\psi\}$ parameter space that are compatible with the measured lifetime ratios. A parameter point is considered allowed if, after varying all input parameters within their uncertainties, the theoretical prediction overlaps with the experimentally measured lifetime ratio at the 1σ level. The dominant theoretical uncertainty originates from the SM HQE prediction for the lifetime ratios, cf. Eq. (4.36). In addition, we vary the b -quark mass within a 10% interval to account for the ambiguity of different mass schemes.

The corresponding predictions for the inclusive BRs are shown in Figures D.15 and D.19. For each value of m_ψ , the branching ratio is evaluated using $\text{BR}^{\text{Th.}} = \Gamma_3^{\text{meso}}(G_X^{f_1 f_2}, m_\psi) \tau(B^+)^{\text{Exp.}}$, where the coupling $G_X^{f_1 f_2}$ is taken at the maximum value allowed by the lifetime-ratio analysis. To obtain a conservative upper bound, the experimental lifetime $\tau(B^+)^{\text{Exp.}}$ and the b -quark mass are both chosen at the upper edge of their respective allowed ranges.

Before discussing the results in detail, it is useful to highlight the two main factors that determine the predicted inclusive BRs when the Mesogenesis couplings are constrained by the lifetime ratio $\tau(B^+)/\tau(B_d)$:

1. **Topology dependence.** The strength of the lifetime-ratio constraint depends on the topology through which a given operator contributes (see Figure 4.1). For example, operators that contribute only to one of the two mesons, or

Operator	Affects B^+ ?	Affects B_d ?	Topology	BR ($\tau(B^+)/\tau(B_d)$)
$(\bar{b}u^c)(\bar{\psi}d^c)$	Yes	Yes	IV(B^+), III(B_d)	Figure D.18 (a)
$(\bar{b}c^c)(\bar{\psi}d^c)$	No	Yes	III	Figure D.18 (b)
$(\bar{d}u^c)(\bar{\psi}b^c)$	Yes	Yes	III both	unconstrained
$(\bar{d}c^c)(\bar{\psi}b^c)$	No	Yes	III	Figure D.18 (c)
$(\bar{\psi}d^c)(\bar{\psi}b^c)$	No	Yes	I	Figure D.18 (d)
$(\bar{b}u^c)(\bar{u}d^c)$	Yes	Yes	{IV, I}(B^+), I(B_d)	Figure D.18 (e)
$(\bar{b}c^c)(\bar{c}d^c)$	No	Yes	I	Figure D.18 (f)
$(\bar{b}c^c)(\bar{u}d^c)$	Yes	Yes	I both	unconstrained
$(\bar{b}u^c)(\bar{c}d^c)$	Yes	Yes	IV(B^+), I(B_d)	Figure D.19 (a)
$(\bar{b}u^c)(\bar{\psi}s^c)$	Yes	No	IV	Figure D.19 (b)
$(\bar{\psi}b^c)(\bar{u}s^c)$	Yes	No	III	Figure D.19 (c)
$(\bar{b}u^c)(\bar{u}s^c)$	Yes	No	{IV, I}	Figure D.19 (d)
$(\bar{b}u^c)(\bar{c}s^c)$	Yes	No	IV	Figure D.19 (e)
$(\bar{b}c^c)(\bar{u}s^c)$	Yes	No	I	Figure D.19 (f)

Table 4.2: $\mathcal{L}_{-1/3}$ operator contributions to B^+ and B_d meson decay rates from dimension-6 four-quark operators, showing topologies and inclusive BRs using couplings constrained by $\tau(B^+)/\tau(B_d)$. Rows highlighted in red correspond to the least-constrained operators. Rows highlighted in yellow indicate operators that contribute equally to both B^+ and B_d mesons and thus cannot be constrained by the lifetime ratio.

through different topologies in the B^+ and B_d channels, are generally subject to weaker constraints.

- Phase-space suppression.** The available phase space directly determines the size of the partial decay width. Even if the lifetime-ratio constraint on a coupling is relatively weak, the corresponding branching ratio can remain small when the final state contains heavy particles, such as a charm quark or the dark fermion ψ .

The interplay of these two effects governs the resulting BRs. In particular, weakly constrained operators may still predict small branching ratios because of phase-space suppression, whereas operators involving light final-state particles can give comparatively large branching ratios even when their couplings are more tightly constrained.

From the inclusive BR analysis, we identify a broad class of Mesogenesis operators for which the predicted BR exceeds the acceptable upper limit of approximately 5% when the couplings are constrained using the lifetime ratio $\tau(B^+)/\tau(B_d)$ alone. This behaviour is most pronounced for operators contributing exclusively to either

the B^+ or the B_d lifetime, in particular through topology III with at most one massive final-state particle or through topology I involving only light quarks (u , d , and s). The corresponding operator classes, which are only weakly constrained by the lifetime-ratio analysis, are highlighted in red in [Tables 4.1](#) and [4.2](#) for the $\mathcal{L}_{2/3}$ and $\mathcal{L}_{-1/3}$ scenarios, respectively.

We further find a small number of operators whose predicted BRs remain above the 5% level even after imposing the collider constraints. However, these deviations are well within the 10% theoretical uncertainty assigned to the inclusive branching-ratio calculation and therefore do not provide additional exclusions beyond the existing collider bounds.

We also identify a class of operators whose contributions to the B^+ and B_d decay widths are identical, such that they cancel exactly in the lifetime ratio $\tau(B^+)/\tau(B_d)$. Consequently, the lifetime ratio does not constrain these operators. This occurs for operators of the form $(du^c)(fb^c)$, with $f = \{c, \psi\}$, whose contributions arise through the same topology in both decay channels. These operators are highlighted in yellow in [Table 4.2](#).

It is important to note that this cancellation relies on the presence of the charge-conjugated fields. Operators with the same quark flavour content but a different Lorentz structure, such as $(du)(cb)$, do not exhibit this behaviour. Instead, they contribute to the B^+ and B_d lifetimes through different topologies, so that no cancellation occurs in the lifetime ratio (see, e.g., Ref. [\[81\]](#)).

4.3 Exclusive Calculations: LCSR

Having discussed the inclusive Mesogenesis analysis within the HQE, we now turn to the corresponding exclusive decay channels. In this section, we study the meson-to-baryon decay $B^+ \rightarrow p\psi$, which constitutes the simplest baryonic transition allowed in the Mesogenesis framework, using the method of LCSR introduced in [Section 1.4](#).

The $B \rightarrow p$ transition has previously been studied within the framework of LCSR using proton DAs up to twist-six [\[95, 106\]](#). In the present work, however, we employ the complementary formulation based on B -meson DAs. One motivation for this approach is that higher-twist corrections from the B -meson DAs can be incorporated systematically and directly within the sum-rule framework. In addition, the uncertainties associated with the B -meson DAs are expected to be smaller than those arising from the less well-constrained proton DAs. The present analysis therefore also provides an important cross-check between the two different LCSR formulations. A detailed comparison between the two approaches will be presented later in this section.

For the exclusive analysis, it is convenient to rewrite the effective Hamiltonian introduced in [Eq. \(4.2\)](#) in the compact notation used in [\[95\]](#). Restricting ourselves to the operators proportional to $G_{(d)}$ and $G_{(b)}$, relevant for the $B^+ \rightarrow p\psi$ transition, we write

$$\mathcal{H}_{(-1/3)} = -G_{(d)}\bar{\mathcal{O}}_{(d)}\Psi^c - G_{(d)}^*\bar{\Psi}^c\mathcal{O}_{(d)} + \{d \leftrightarrow b\}, \quad (4.39)$$

where the shorthand $\{d \leftrightarrow b\}$ denotes the analogous contributions proportional to $G_{(b)}$. The couplings $G_{(d)}$ and $G_{(b)}$ correspond directly to the Wilson coefficients $C_\psi^{(ud)}$

and $\tilde{C}_\psi^{(ud)}$ introduced in Eq. (4.3), respectively. The local three-quark operators entering Eq. (4.39) are defined as

$$\overline{\mathcal{O}}_{(d)} = i\epsilon_{ijk} (\bar{u}_R^i b_R^{cj}) \bar{d}_R^k, \quad \mathcal{O}_{(d)} = i\epsilon_{ijk} d_R^i (\bar{b}_R^{cj} u_R^k). \quad (4.40)$$

where the superscript c denotes charge-conjugated fields and $q_R = \frac{1}{2}(1 + \gamma_5)q$. The indices i, j, k label the colour degrees of freedom. The corresponding operators for model (b) are obtained through the replacement $d \leftrightarrow b$:

$$\overline{\mathcal{O}}_{(b)} = i\epsilon_{ijk} (\bar{u}_R^i d_R^{cj}) \bar{b}_R^k, \quad \mathcal{O}_{(b)} = i\epsilon_{ijk} b_R^i (\bar{d}_R^{cj} u_R^k). \quad (4.41)$$

The decay amplitude for the process $B^+(p_B) \rightarrow p(P) \psi(q)$ is expressed as

$$\mathcal{A}^{(X)}(B^+ \rightarrow p \psi) = G^{(X)} \langle p(P) | \mathcal{O}^{(X)} | B^+(p_B) \rangle u_\psi^c(q), \quad (4.42)$$

where $u_\psi^c(q)$ denotes the charge-conjugated spinor of the outgoing dark-sector fermion ψ with momentum q , and $X = \{d, b\}$ labels the Mesogenesis model. The hadronic matrix element is parameterised in terms of four independent $B \rightarrow p$ form factors:

$$\begin{aligned} \langle p(P) | \overline{\mathcal{O}}_{(X)} | B^+(P+q) \rangle &= F_{B \rightarrow p_R}^{(X)}(q^2) \bar{u}_{pR}(P) + F_{B \rightarrow p_L}^{(X)}(q^2) \bar{u}_{pL}(P) \\ &\quad + \tilde{F}_{B \rightarrow p_R}^{(X)}(q^2) \bar{u}_{pR}(P) \frac{\not{q}}{m_p} + \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) \bar{u}_{pL}(P) \frac{\not{q}}{m_p}, \end{aligned} \quad (4.43)$$

where $q = p_B - P$ denotes the momentum transfer, and $\bar{u}_{pR}(P)$ ($\bar{u}_{pL}(P)$) denotes the right-handed (left-handed) proton Dirac spinor with momentum P . The above decomposition is dictated by Lorentz invariance together with the quantum numbers of the external states and the operator. The initial B meson is a pseudoscalar state and therefore contributes only through its momentum, while the final proton state provides the Dirac spinor $u_p(P)$. Since the operator is chiral, the matrix element can be expanded in terms of independent Dirac structures constructed from the chiral projectors $P_{R,L}$ and the available four-momenta. Using momentum conservation, $p_B = P + q$, one may equivalently choose either $\not{p}_B$ or $\not{q}$ as the momentum-dependent Dirac structure. The use of $\not{q}$ is therefore merely a choice of basis. Note that structures proportional to $\not{P}$ are not independent and can be reduced using the proton equation of motion, $\not{P} u_p(P) = m_p u_p(P)$. The factor $1/m_p$ is introduced in the second line such that all form factors have the same mass dimension.

The four form factors $F_{B \rightarrow p_R}^{(X)}(q^2)$, $F_{B \rightarrow p_L}^{(X)}(q^2)$, $\tilde{F}_{B \rightarrow p_R}^{(X)}(q^2)$, and $\tilde{F}_{B \rightarrow p_L}^{(X)}(q^2)$ encode the non-perturbative QCD dynamics of the baryonic transition and constitute the main quantities to be determined from LCSR⁶.

To derive the LCSR for the form factors introduced above, we consider the following vacuum-to- B correlation function:

$$\mathcal{F}^{(X)}(P, q) = i \int d^4x e^{iP \cdot x} \langle 0 | T \{ \eta_p(x) \overline{\mathcal{O}}_{(X)}(0) \} | B^+(P+q) \rangle, \quad (4.44)$$

⁶Since the operators $\overline{\mathcal{O}}_{(X)}$ contain the chiral structure $\bar{d} P_L$, the matrix element inherits the same chirality. Multiplying the decomposition from the right by P_R , one finds that the structures proportional to $\bar{u}_{pL}(P)$ and $\bar{u}_{pR}(P) \not{q}$ vanish, implying $F_{B \rightarrow p_L}^{(X)}(q^2) = 0$ and $\tilde{F}_{B \rightarrow p_R}^{(X)}(q^2) = 0$. Nevertheless, for generality we retain the full decomposition in Eq. (4.43).

where η_p denotes the interpolating current with the quantum numbers of the proton. The correlation function depends on the two independent momenta P and q , while the external B meson carries momentum $P + q$.

The choice of the interpolating current is an important ingredient in the construction of baryonic sum rules. In general, several local three-quark currents with the quantum numbers of the proton can be constructed, differing in their Dirac structure and in their overlap with the proton ground state and excited baryonic resonances (see e.g. [107] for details). In the present work, we adopt the Ioffe current [108], which corresponds to a particular linear combination of two other three-quark currents known to provide a strong coupling to the proton ground state and stable sum rules in the nucleon channel. The proton interpolating current is therefore chosen as

$$\eta_p(x) = \epsilon_{ijk} [u^{Ti}(x) C \gamma^\mu u^j(x)] \gamma_5 \gamma_\mu d^k(x), \quad (4.45)$$

where the colour indices are shown explicitly and C denotes the charge-conjugation matrix, whose convention is specified in Eqs. (A.16) and (A.17).

The correlation function Eq. (4.44) is a Dirac object depending on the two independent four-momenta P and q . In order to derive separate sum rules for the form factors appearing in Eq. (4.43), it is convenient to decompose the correlation function into a basis of independent Lorentz and Dirac structures:

$$\begin{aligned} \mathcal{F}^{(X)}(P, q) = & \left[\mathcal{F}_1^{(X)} + \mathcal{F}_2^{(X)} \frac{\not{P}}{m_B} + \mathcal{F}_3^{(X)} \frac{\not{q}}{m_B} + \mathcal{F}_4^{(X)} \frac{\not{P}\not{q}}{m_B^2} \right] P_L \\ & + \left[\mathcal{F}_5^{(X)} + \mathcal{F}_6^{(X)} \frac{\not{P}}{m_B} + \mathcal{F}_7^{(X)} \frac{\not{q}}{m_B} + \mathcal{F}_8^{(X)} \frac{\not{P}\not{q}}{m_B^2} \right] P_R, \end{aligned} \quad (4.46)$$

The scalar coefficient functions $\mathcal{F}_{1,\dots,8}^{(X)}$ are Lorentz-invariant amplitudes depending only on the kinematic invariants P^2 and q^2 ; this dependence is suppressed in Eq. (4.46) for convenience. Matching these amplitudes between the hadronic and QCD representations yields the desired LCSR relations.

Employing the unitarity relation, we derive the hadronic representation of the correlation function. In the timelike region $P^2 > 0$, its imaginary part can be expressed as

$$\begin{aligned} \frac{1}{\pi} \text{Im } \mathcal{F}^{(X)}(P, q) = & \delta(P^2 - m_p^2) \langle 0 | \eta_p | \overline{p(P)} \rangle \langle \overline{p(P)} | \overline{\mathcal{O}_{(X)}} | B^+(P + q) \rangle \\ & + \delta(P^2 - m_{N^*}^2) \langle 0 | \eta_p | \overline{N^*(P)} \rangle \langle \overline{N^*(P)} | \overline{\mathcal{O}_{(X)}} | B^+(P + q) \rangle \\ & + \rho_h^{(X)}(P, q), \end{aligned} \quad (4.47)$$

where the overlines indicate summation over baryon polarizations. The inserted intermediate states are required to carry the same conserved quantum numbers as the interpolating current η_p , namely baryon number, isospin, electric charge, and spin. The current in Eq. (4.45) has non-vanishing overlap with both positive- and negative-parity spin-1/2 baryons. Consequently, in addition to the proton contribution, we explicitly isolate the contribution of $N^* \equiv N(1535)$, the lowest nucleon resonance with spin-parity $J^P = 1/2^-$ [96]. Resonances and continuum states with quantum

numbers $J^P = 1/2^+$ ($J^P = 1/2^-$) and masses above the proton (the N^*) are included in the spectral density $\rho_h^{(X)}(P, q)$.

Analogously to Eq. (4.43), the matrix element involving the negative-parity N^* state is parameterized as

$$\begin{aligned} \langle N^*(P) | \bar{\mathcal{O}}_{(X)} | B^+(P+q) \rangle = & \left[F_{B \rightarrow N_R^*}^{(X)}(q^2) \bar{u}_{N^*R}(P) + F_{B \rightarrow N_L^*}^{(X)}(q^2) \bar{u}_{N^*L}(P) \right] \gamma_5 \\ & + \left[\tilde{F}_{B \rightarrow N_R^*}^{(X)}(q^2) \bar{u}_{N^*R}(P) + \tilde{F}_{B \rightarrow N_L^*}^{(X)}(q^2) \bar{u}_{N^*L}(P) \right] \gamma_5 \frac{\not{q}}{m_{N^*}}. \end{aligned} \quad (4.48)$$

The couplings of the interpolating current to the proton and to the lowest negative-parity nucleon resonance N^* are defined as

$$\langle 0 | \eta_p | p(P) \rangle = \lambda_p m_p u_p(P), \quad \langle 0 | \eta_p | N^*(P) \rangle = \lambda_{N^*} m_{N^*} \gamma_5 u_{N^*}(P), \quad (4.49)$$

where $u_p(P)$ and $u_{N^*}(P)$ denote the Dirac bispinors of the proton and N^* states with four-momentum P , respectively, while λ_p and λ_{N^*} are the corresponding decay constants describing the overlap of the interpolating current with the physical baryon states. The additional γ_5 in the second matrix element accounts for the negative parity of the N^* state.

Substituting Eqs. (4.43), (4.48) and (4.49) into Eq. (4.47), and using the spin-sum relations

$$\sum_s u_p(P, s) \bar{u}_p(P, s) = \not{P} + m_p, \quad \sum_s u_{N^*}(P, s) \bar{u}_{N^*}(P, s) = \not{P} + m_{N^*},$$

we obtain the hadronic representation of the spectral density:

$$\begin{aligned} \frac{1}{\pi} \text{Im } \mathcal{F}^{(X)}(P, q) = & \delta(P^2 - m_p^2) \lambda_p m_p \left\{ (\not{P} + m_p) \left[F_{B \rightarrow p_R}^{(X)}(q^2) P_L + F_{B \rightarrow p_L}^{(X)}(q^2) P_R \right] \right. \\ & \left. + (\not{P} + m_p) \frac{\not{q}}{m_p} \left[\tilde{F}_{B \rightarrow p_R}^{(X)}(q^2) P_R + \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) P_L \right] \right\} \\ & - \delta(P^2 - m_{N^*}^2) \lambda_{N^*} m_{N^*} \left\{ (\not{P} - m_{N^*}) \left[F_{B \rightarrow N_R^*}^{(X)}(q^2) P_L + F_{B \rightarrow N_L^*}^{(X)}(q^2) P_R \right] \right. \\ & \left. + (\not{P} - m_{N^*}) \frac{\not{q}}{m_{N^*}} \left[\tilde{F}_{B \rightarrow N_R^*}^{(X)}(q^2) P_R + \tilde{F}_{B \rightarrow N_L^*}^{(X)}(q^2) P_L \right] \right\} \\ & + \rho_h^{(X)}(P, q). \end{aligned} \quad (4.50)$$

The relative minus sign in the N^* contribution originates from the γ_5 factors in Eqs. (4.48) and (4.49), using $\gamma_5(\not{P} + m_{N^*})\gamma_5 = -\not{P} + m_{N^*}$.

More generally, the hadronic spectral density can itself be decomposed into the same basis of independent Lorentz and Dirac structures:

$$\begin{aligned}\rho^{(X)}(P, q) = & \left[\rho_1^{(X)} + \rho_2^{(X)} \frac{\not{P}}{m_B} + \rho_3^{(X)} \frac{\not{q}}{m_B} + \rho_4^{(X)} \frac{\not{P}\not{q}}{m_B^2} \right] P_L \\ & + \left[\rho_5^{(X)} + \rho_6^{(X)} \frac{\not{P}}{m_B} + \rho_7^{(X)} \frac{\not{q}}{m_B} + \rho_8^{(X)} \frac{\not{P}\not{q}}{m_B^2} \right] P_R,\end{aligned}\quad (4.51)$$

where the dependence of the invariant functions $\rho_i^{(X)}$ on the kinematic variables (P^2, q^2) has been suppressed for brevity. As explained in [Section 1.4](#), the imaginary part of the correlation function enters a dispersion relation in the variable $P^2 \rightarrow s$, which is valid for all values of P^2 away from the physical singularities at fixed q^2 :

$$\mathcal{F}^{(X)}(P, q) = \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\text{Im } \mathcal{F}^{(X)}(s, q)}{s - P^2}, \quad (4.52)$$

where the lower integration limit s_{th} is determined by the physical threshold in the channel interpolated by the current η_p . In the present case, the lowest state carrying the proton quantum numbers is the proton itself, such that $s_{\text{th}} = m_p^2$.

Using the dispersion relation in [Eq. \(4.52\)](#) and comparing the hadronic representation in [Eq. \(4.50\)](#) with the general decomposition in [Eq. \(4.46\)](#), one obtains dispersion relations for the invariant amplitudes by matching the corresponding Lorentz and Dirac structures. For the first four amplitudes, the resulting relations read

$$\mathcal{F}_1^{(X)}(P^2, q^2) = \frac{\lambda_p m_p^2}{m_p^2 - P^2} F_{B \rightarrow p_R}^{(X)}(q^2) + \frac{\lambda_{N^*} m_{N^*}^2}{m_{N^*}^2 - P^2} F_{B \rightarrow N_R^*}^{(X)}(q^2) + \int_{s_h}^{\infty} ds \frac{\rho_1^{(X)}(s, q^2)}{s - P^2}, \quad (4.53)$$

$$\mathcal{F}_2^{(X)}(P^2, q^2) = \frac{\lambda_p m_p m_B}{m_p^2 - P^2} F_{B \rightarrow p_R}^{(X)}(q^2) - \frac{\lambda_{N^*} m_{N^*} m_B}{m_{N^*}^2 - P^2} F_{B \rightarrow N_R^*}^{(X)}(q^2) + \int_{s_h}^{\infty} ds \frac{\rho_2^{(X)}(s, q^2)}{s - P^2}, \quad (4.54)$$

$$\mathcal{F}_3^{(X)}(P^2, q^2) = \frac{\lambda_p m_p m_B}{m_p^2 - P^2} \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) + \frac{\lambda_{N^*} m_{N^*} m_B}{m_{N^*}^2 - P^2} \tilde{F}_{B \rightarrow N_L^*}^{(X)}(q^2) + \int_{s_h}^{\infty} ds \frac{\rho_3^{(X)}(s, q^2)}{s - P^2}, \quad (4.55)$$

$$\mathcal{F}_4^{(X)}(P^2, q^2) = \frac{\lambda_p m_B^2}{m_p^2 - P^2} \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) - \frac{\lambda_{N^*} m_B^2}{m_{N^*}^2 - P^2} \tilde{F}_{B \rightarrow N_L^*}^{(X)}(q^2) + \int_{s_h}^{\infty} ds \frac{\rho_4^{(X)}(s, q^2)}{s - P^2}. \quad (4.56)$$

The remaining four relations for $\mathcal{F}_{5,\dots,8}^{(X)}$ are obtained from the above expressions by interchanging the chiral labels, $R \leftrightarrow L$, together with the replacements $1 \rightarrow 5, 2 \rightarrow 6, 3 \rightarrow 7, 4 \rightarrow 8$. Here s_h denotes the physical continuum threshold in the proton channel. Since the lightest multi-particle state carrying the same quantum numbers as the proton current is the $p\pi$ state, one has $s_h = (m_p + m_\pi)^2$ ⁷.

⁷In contrast to the proton and N^* contributions, the $p\pi$ state is a two-particle state whose

The relations derived above constitute the hadronic representation of the correlation function. The same correlation function can also be evaluated in QCD using the light-cone OPE, which is valid in the spacelike region $P^2 \ll 0$, where the virtualities are sufficiently far away from the physical hadronic poles and cuts. In this region, the correlation function admits the dispersion representation

$$\mathcal{F}_{\text{OPE}}^{(X)}(P, q) = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im} \mathcal{F}_{\text{OPE}}^{(X)}(s, q)}{s - P^2}, \quad (4.57)$$

where the spectral density is calculated directly from quark and gluon degrees of freedom. The lower integration limit reflects the approximation $m_u = m_d = 0$. We now invoke the quark–hadron duality approximation to model the higher resonance and continuum contributions appearing in Eqs. (4.53) to (4.56). Specifically, one assumes that above a certain effective threshold s_0 , the hadronic spectral densities are well approximated by the corresponding OPE spectral densities:

$$\int_{s_h}^\infty ds \frac{\rho_n^{(X)}(s, q^2)}{s - P^2} = \frac{1}{\pi} \int_{s_0^{(n)}}^\infty ds \frac{\text{Im} \mathcal{F}_n^{(X)\text{OPE}}(s, q^2)}{s - P^2}, \quad (n = 1, \dots, 8), \quad (4.58)$$

Here $s_0^{(n)}$ denotes the effective continuum-threshold parameter associated with the invariant amplitude $\mathcal{F}_n^{(X)}$. It is important to distinguish the physical threshold s_h from the effective continuum thresholds $s_0^{(n)}$. The threshold s_h is fixed by the physical hadronic spectrum and is therefore universal for all invariant amplitudes. In contrast, the parameters $s_0^{(n)}$ arise from the quark–hadron duality approximation and effectively model the transition between explicitly isolated low-lying states and the higher-state continuum. Since different invariant amplitudes are associated with different Dirac structures and may couple differently to excited states, the corresponding effective thresholds need not be identical.

Equating the hadronic and OPE representations of the invariant amplitudes, implementing the quark–hadron duality approximation for the higher-state contributions, and subsequently applying a Borel transformation in the variable $P^2 \rightarrow M^2$, one arrives at a system of eight LCSR relations for the $B \rightarrow p$ and $B \rightarrow N^*$ transition form factors. The first four sum rules are given by

$$\begin{aligned} \lambda_p m_p^2 e^{-m_p^2/M^2} F_{B \rightarrow p_R}^{(X)}(q^2) + \lambda_{N^*} m_{N^*}^2 e^{-m_{N^*}^2/M^2} F_{B \rightarrow N_R^*}^{(X)}(q^2) \\ = \frac{1}{\pi} \int_0^{s_0^{(1)}} ds e^{-s/M^2} \text{Im} \mathcal{F}_1^{(X)\text{OPE}}(s, q^2), \end{aligned} \quad (4.59)$$

$$\begin{aligned} \lambda_p m_p m_B e^{-m_p^2/M^2} F_{B \rightarrow p_R}^{(X)}(q^2) - \lambda_{N^*} m_{N^*} m_B e^{-m_{N^*}^2/M^2} F_{B \rightarrow N_R^*}^{(X)}(q^2) \\ = \frac{1}{\pi} \int_0^{s_0^{(2)}} ds e^{-s/M^2} \text{Im} \mathcal{F}_2^{(X)\text{OPE}}(s, q^2), \end{aligned} \quad (4.60)$$

invariant mass is not fixed. It therefore generates a continuous spectral contribution (branch cut) starting at $s = (m_p + m_\pi)^2$, rather than an isolated pole in the dispersion relation.

$$\begin{aligned} \lambda_p m_p m_B e^{-m_p^2/M^2} \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) + \lambda_{N^*} m_{N^*} m_B e^{-m_{N^*}^2/M^2} \tilde{F}_{B \rightarrow N_L^*}^{(X)}(q^2) \\ = \frac{1}{\pi} \int_0^{s_0^{(4)}} ds e^{-s/M^2} \text{Im } \mathcal{F}_3^{(X)\text{OPE}}(s, q^2), \end{aligned} \quad (4.61)$$

$$\begin{aligned} \lambda_p m_B^2 e^{-m_p^2/M^2} \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) - \lambda_{N^*} m_B^2 e^{-m_{N^*}^2/M^2} \tilde{F}_{B \rightarrow N_L^*}^{(X)}(q^2) \\ = \frac{1}{\pi} \int_0^{s_0^{(4)}} ds e^{-s/M^2} \text{Im } \mathcal{F}_4^{(X)\text{OPE}}(s, q^2). \end{aligned} \quad (4.62)$$

The remaining four sum rules corresponding to $\mathcal{F}_{5,\dots,8}^{(X)}$ are obtained analogously from the relations below Eq. (4.56). The sum rules for the physical $B \rightarrow p$ form factors are obtained by constructing suitable linear combinations of the LCSR relations such that the unwanted $B \rightarrow N^*$ contributions cancel. One finds

$$F_{B \rightarrow p_R}^{(X)}(q^2) = \frac{e^{m_p^2/M^2}}{\lambda_p m_p (m_p + m_{N^*})} \left[I_1^{(X)}(M^2, q^2) + \frac{m_{N^*}}{m_B} I_2^{(X)}(M^2, q^2) \right], \quad (4.63)$$

and

$$\tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) = \frac{e^{m_p^2/M^2}}{\lambda_p m_B (m_p + m_{N^*})} \left[I_3^{(X)}(M^2, q^2) + \frac{m_{N^*}}{m_B} I_4^{(X)}(M^2, q^2) \right], \quad (4.64)$$

where we introduced the shorthand notation

$$I_n^{(X)}(M^2, q^2) \equiv \frac{1}{\pi} \int_0^{s_{0,n}} ds e^{-s/M^2} \text{Im } \mathcal{F}_n^{(X)\text{OPE}}(s, q^2). \quad (4.65)$$

The corresponding sum rules for the $B \rightarrow N^*$ transition form factors are obtained analogously by eliminating the proton contributions instead.

Before turning to the explicit OPE calculation, it is important to comment on an additional subtlety in the hadronic representation. Between the proton ground state and the negative-parity resonance $N^*(1535)$, there exists the broad positive-parity Roper resonance $N(1440)$ with quantum numbers $J^P = 1/2^+$ [96]. In principle, this state should also be taken into account explicitly in the sum rules by isolating its pole contribution in the same manner as for the proton and N^* states.

In practice, however, the treatment of the Roper resonance is considerably more complicated. Owing to its large decay width and its strong overlap with the nearby multi-particle continuum, the Roper state does not manifest itself as a narrow isolated resonance. Moreover, introducing an explicit Roper contribution would require additional hadronic parameters, in particular its decay constant and transition form factors, for which no reliable estimates currently exist. The interpolating current furthermore couples simultaneously to the proton, the Roper resonance, and other positive-parity excitations, rendering a clean separation numerically difficult.

For these reasons, we adopt a simplified hadronic ansatz following previous LCSR analyses of nucleon electromagnetic form factors (see e.g. [109, 110]). In those

studies, the same proton interpolating current was employed, while the $N(1440)$ and $N^*(1535)$ resonances were effectively included in the spectral densities through the quark–hadron duality approximation in Eq. (4.58). Following the same strategy, we absorb the Roper resonance contribution into the higher-resonance and continuum part modeled through the quark–hadron duality approximation. In addition, we adopt a common effective continuum threshold for all invariant amplitudes, $s_0^{(n)} \equiv s_0$. These assumptions considerably simplify the resulting sum rules. Furthermore, as will be shown below, all invariant amplitudes except $\mathcal{F}_1^{(X)}$ and $\mathcal{F}_4^{(X)}$ vanish in the adopted approximation. Consequently,

$$F_{B \rightarrow p_R}^{(X)}(q^2) = \frac{1}{\pi \lambda_p m_p^2} \int_0^{s_0} ds e^{\frac{m_p^2 - s}{M^2}} \text{Im } \mathcal{F}_1^{(X)\text{OPE}}(s, q^2), \quad (4.66)$$

and

$$\tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) = \frac{1}{\pi \lambda_p m_B^2} \int_0^{s_0} ds e^{\frac{m_p^2 - s}{M^2}} \text{Im } \mathcal{F}_4^{(X)\text{OPE}}(s, q^2). \quad (4.67)$$

Having established the hadronic representation and the corresponding LCSR relations, we now turn to the explicit calculation of the OPE side of the correlation function. For definiteness, we focus in the following on model (d) and consider the correlation function

$$\mathcal{F}^{(d)}(P, q) = i \int d^4x e^{iP \cdot x} \langle 0 | T \{ \eta_p(x) \bar{\mathcal{O}}_{(d)}(0) \} | B^+(P + q) \rangle. \quad (4.68)$$

The extension to model (b) proceeds analogously. The OPE receives contributions from both the two-particle and three-particle B -meson DAs. The corresponding leading-order diagrams are shown in Figure 4.5. The left diagram corresponds to the two-particle DA contribution, while the right diagram represents the three-particle DA contribution generated by soft-gluon emission from the light-quark propagator.

The OPE calculation proceeds by contracting the time-ordered product in Eq. (4.68). We first consider the contribution of the two-particle B -meson DAs corresponding to the left diagram in Figure 4.5. At leading order, the u - and d -quark fields are contracted using the free light-quark propagator. We find it most convenient to perform the calculation entirely in coordinate space. After contracting the colour indices, simplifying the resulting Dirac structures with the charge-conjugation identities in Eqs. (A.16) and (A.17), and taking into account the two possible contractions of the u -quark fields, which yield identical contributions, we obtain

$$\mathcal{F}^{(d)}(P, q) = \int d^4x e^{iP \cdot x} \frac{-4 x_\alpha x_\beta}{[(2\pi^2)(x^2)^2]^2} \langle 0 | \bar{b}(0) P_L \gamma^\alpha \gamma^\mu u(x) | B^+(P + q) \rangle \gamma_5 \gamma_\mu \gamma^\beta P_L. \quad (4.69)$$

The remaining nonlocal vacuum-to- B matrix element is subsequently expanded in terms of the B -meson light-cone DAs. To this end, we employ the HQET framework,

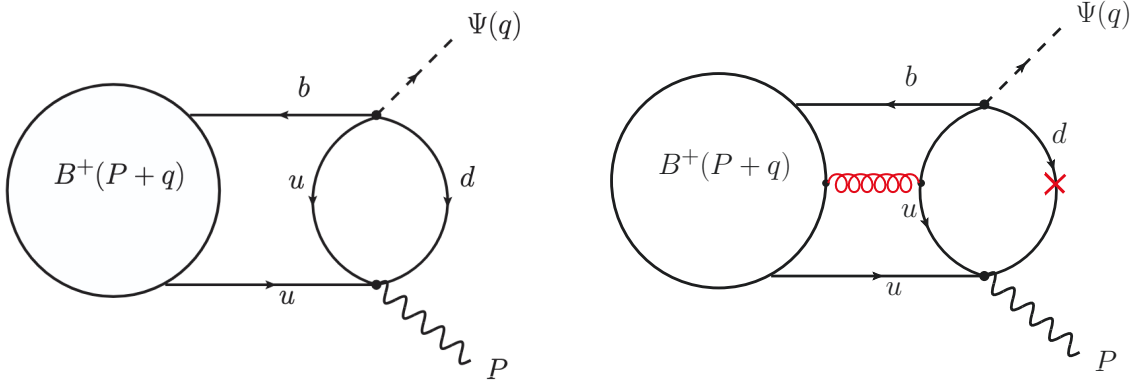


Figure 4.5: Leading-order contributions to the correlation function in the LCSR approach with B -meson DAs. Left: contribution from the two-particle DAs. Right: diagram describing the contribution of the three-particle (quark-antiquark-gluon) B -meson DAs. The curly line represents the gluon component of the DA, while the cross indicates the second diagram in which the gluon is emitted from the d -quark line.

in which the heavy b -quark field is decomposed as⁸

$$b(x) = e^{-im_b v \cdot x} h_v(x),$$

where v^μ denotes the four-velocity of the B meson, $v^\mu = (P+q)^\mu/m_B$, and h_v is the effective heavy-quark field in HQET. This decomposition factors out the dominant mechanical momentum $m_b v^\mu$ carried by the heavy quark, while the residual soft dynamics are encoded in the slowly varying field h_v . The resulting nonlocal matrix elements of the form

$$\langle 0 | \bar{h}_v(0) \Gamma u(x) | \bar{B}(v) \rangle$$

are then parameterized in terms of the two-particle B -meson DAs, whose definitions and conventions are collected in [Section D.3](#).

Substituting the decomposition of the nonlocal matrix element in terms of the B -meson DAs from [Section D.3](#) into [Eq. \(4.69\)](#), we arrive at the following intermediate expression for the OPE representation of the correlation function:

$$\begin{aligned} \mathcal{F}^{(d)\text{OPE}}(P, q) = & \left(\frac{if_B m_B}{4} \right) \int_0^\infty d\omega \int d^4x e^{i(P-\omega v) \cdot x} \frac{x_\alpha x_\beta}{[(2\pi^2)(x^2)^2]^2} \\ & \times \left\{ -\text{Tr}[\gamma^\mu \gamma^\alpha (1 + \not{v})] \phi_+^B(\omega) + \text{Tr}[\gamma^\mu \gamma^\alpha (1 + \not{v}) \not{x}] \frac{i}{2} \Phi_\pm^B(\omega) \right\} \gamma_5 \gamma_\mu \gamma_\beta (1 + \not{v}) + \dots, \end{aligned} \quad (4.70)$$

where for brevity only the contributions of the twist-two and twist-three DAs are shown explicitly. After performing the Dirac algebra and evaluating the coordinate-

⁸At leading order in the heavy-quark expansion, this decomposition is equivalent to [Eq. \(B.2\)](#), with $h_v \equiv b_v$. We retain the notation h_v here to remain consistent with the standard parametrisation of the B -meson light-cone DAs; see [Section D.3](#).

space integrals, we obtain

$$\begin{aligned}\mathcal{F}^{(d)\text{OPE}}(P, q) = & \frac{f_B m_B}{16\pi^2} \int_0^\infty d\omega \left((P - \omega v)^2 \ln \left[- (P - \omega v)^2 \right] \phi_+^B(\omega) \right. \\ & - \ln \left[- (P - \omega v)^2 \right] (\not{P}\not{v} - \omega) \Phi_\pm^B(\omega) - 8 \ln \left[- (P - \omega v)^2 \right] g_+^B(\omega) \\ & \left. + 8 \frac{\not{P}\not{v} - \omega}{(P - \omega v)^2} G_\pm^B(\omega) \right) P_L, \quad (4.71)\end{aligned}$$

where the contributions of the twist-four and twist-five DAs are also included. Note that in Eq. (4.71), we retain only those terms that generate a nonvanishing imaginary part in the variable P^2 . Constant and polynomial contributions in P^2 are omitted, since they disappear after the Borel transformation.

Repeating the same calculation for model (b), we find the simple relation

$$\mathcal{F}^{(b)\text{OPE}}(P, q) = 2 \mathcal{F}^{(d)\text{OPE}}(P, q). \quad (4.72)$$

Consequently, the corresponding sum rules and form factors of model (b) are directly obtained from those of model (d) through an overall factor of two.⁹

We now turn to the calculation of the three-particle DA contributions. These arise from the soft-gluon emission off the light u - and d -quark propagators, corresponding to the right diagram in Figure 4.5. For massless light quarks, we employ the light-cone expansion of the quark propagator in the external gluon field [111]:

$$\begin{aligned}-i\langle 0|T\{q_\alpha^i(x)\bar{q}_\beta^k(0)\}|0\rangle = & \frac{\not{x}_{\alpha\beta}\delta^{ik}}{2\pi^2(-x^2)^2} \\ & + \frac{g_s(\lambda^a)^{ik}}{32\pi^2(-x^2)} \int_0^1 du \left\{ u \sigma_{\mu\nu} G^{a\mu\nu}(ux) \not{x} + \bar{u} \not{x} \sigma_{\mu\nu} G^{a\mu\nu}(ux) \right\}_{\alpha\beta}, \quad (4.73)\end{aligned}$$

where λ^a denote the Gell-Mann matrices, and the Dirac and colour indices are shown explicitly.¹⁰

Using the propagator expansion in Eq. (4.73), following the same colour and Dirac-algebra simplifications as in the two-particle case, and accounting for the second identical contraction of the u -quark fields, we obtain for the diagram in which the soft gluon is emitted from the u -quark propagator the intermediate expression

⁹Interestingly, this simple relation does not arise in the alternative LCSR approach based on proton DAs [95, 106]. This indicates that Eq. (4.72) is a consequence of the specific Dirac and colour structure emerging in the present B -meson-DAs formulation, rather than a fundamental property of the underlying operators themselves.

¹⁰As explained at the end of Section 1.3, the expansion in Eq. (4.73) is equivalent order by order to the quark propagator expansion in the FS gauge discussed in Section 1.3. However, instead of expanding the gluon field around a local point in terms of covariant derivatives, the nonlocal dependence on $G_{\mu\nu}(ux)$ is kept explicitly. This form is more convenient for LCSRs, since it directly matches onto the nonlocal light-cone matrix elements defining the three-particle B -meson DAs.

$$\begin{aligned}
\mathcal{F}^{(d)\text{OPE}}(P, q)|_u &= \frac{1}{16\pi^4} \int_0^1 du \int d^4x e^{i\mathcal{P}\cdot x} \frac{x^{\alpha_1} x^{\alpha_2}}{(-x^2)^3} \\
&\times \left[u \langle 0 | \bar{b}(0) G_{\rho\beta}(ux) P_L \gamma_{\alpha_1} \sigma^{\beta\rho} \gamma_\mu u(x) | B^+(P+q) \rangle \right. \\
&\quad \left. + \bar{u} \langle 0 | \bar{b}(0) G_{\rho\beta}(ux) P_L \sigma^{\beta\rho} \gamma_{\alpha_1} \gamma_\mu u(x) | B^+(P+q) \rangle \right] \gamma_5 \gamma^\mu \gamma_{\alpha_2} P_L,
\end{aligned} \tag{4.74}$$

where $\bar{u} \equiv 1 - u$, and $\mathcal{P} = P - \sigma m_B v$, with the function $\sigma = \sigma(u, \omega_1, \omega_2)$ defined in Eq. (D.30). Similarly, for the diagram in which the soft gluon is emitted from the d -quark propagator, we obtain

$$\begin{aligned}
\mathcal{F}^{(d)\text{OPE}}(P, q)|_d &= -\frac{1}{16\pi^4} \int_0^1 du \int d^4x e^{i\mathcal{P}\cdot x} \frac{x^{\alpha_1} x^{\alpha_2}}{(-x^2)^3} \left[\gamma_5 \gamma_\mu (u \sigma^{\rho\beta} \gamma_{\alpha_1} + \bar{u} \gamma_{\alpha_1} \sigma^{\rho\beta}) P_L \right] \\
&\times \langle 0 | \bar{b}(0) G_{\rho\beta}(ux) P_L \gamma_{\alpha_2} \gamma^\mu u(x) | B^+(P+q) \rangle.
\end{aligned} \tag{4.75}$$

The resulting matrix elements containing two quark fields and one gluon field-strength tensor, appearing in Eqs. (4.74) and (4.75), are then expanded in terms of the three-particle B -meson DAs using the decomposition in Eq. (D.29). Finally, for the diagram with gluon emission from the u -quark propagator, we obtain

$$\mathcal{F}^{(d)\text{OPE}}(P, q)|_u = -\frac{f_B m_B}{64\pi^2} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^1 \frac{du}{\mathcal{P}^2} [u \mathcal{C}(\mathcal{P}, v) + \bar{u} \bar{\mathcal{C}}(\mathcal{P}, v)] P_L, \tag{4.76}$$

The expressions for the coefficient functions $\mathcal{C}$ and $\bar{\mathcal{C}}$ are collected in Section D.4.2.

Evaluating the remaining diagrams, we further obtain

$$\mathcal{F}^{(d)\text{OPE}}(P, q)|_d = -\mathcal{F}^{(d)\text{OPE}}(P, q)|_u, \tag{4.77}$$

$$\mathcal{F}^{(b)\text{OPE}}(P, q)|_d = \mathcal{F}^{(b)\text{OPE}}(P, q)|_u = 0. \tag{4.78}$$

Therefore, the net contribution of the three-particle B -meson DAs to the correlation functions of models (d) and (b) vanishes identically.

The vanishing of the three-particle DA contribution is highly nontrivial and can be traced back to several ingredients. 1. **Interpolating current:** The specific form of the proton interpolating current plays a crucial role. In particular, the Ioffe current is given by a fixed linear combination of two independent three-quark currents (see e.g. [107]). We verified explicitly that each of these currents separately produces nonvanishing three-particle DA contributions, whereas the cancellation occurs only after combining them into the particular Ioffe-current structure. 2. **Operator structure:** The Lorentz structure of the effective interactions in Eqs. (4.40) and (4.41) is equally important. In the present setup, both the diquark bilinear and the quark- Ψ bilinear form Lorentz scalars. Modifying this structure changes the cancellation pattern. For example, considering an interaction of the form $(\bar{u}_R^i \gamma_\rho b_R^c) (\bar{d}_R \gamma^\rho \Psi)$, or the analogous replacement $d \leftrightarrow b$ in model (b), leads to nonvanishing contributions from the soft-gluon emission diagrams in Figure 4.5, even in combination with the

Ioffe current. **3. Quark-mass effects:** As an additional consistency check, we repeated the calculation in momentum space using quark propagators in dimensional regularisation. For massless light quarks, we reproduced the same relations in Eqs. (4.77) and (4.78). On the other hand, introducing a finite strange-quark mass, corresponding for instance to a transition of the type $B \rightarrow \Lambda \Psi$, generates nonvanishing three-particle DA contributions proportional to m_s .

We can now match the OPE result in Eq. (4.71) to the general decomposition in Eq. (4.46). Using $\not{P}\not{\psi} = (P^2 + \not{P}\not{q})/m_B$, we see that the OPE contains only the scalar structure and the $\not{P}\not{q}$ structure multiplying P_L . Therefore, only the invariant amplitudes $\mathcal{F}_1^{(d)\text{OPE}}$ and $\mathcal{F}_4^{(d)\text{OPE}}$ are nonzero, while all remaining amplitudes vanish: $\mathcal{F}_{2,3,5,6,7,8}^{(d)\text{OPE}} = 0$. The nonvanishing invariant amplitudes are

$$\mathcal{F}_1^{(d)\text{OPE}}(P^2, q^2) = \frac{f_B}{16\pi^2} \int_0^\infty d\omega \left[m_B \Delta_\omega \ln(-\Delta_\omega) \phi_+^B(\omega) - \ln(-\Delta_\omega) (P^2 - m_B \omega) \Phi_\pm^B(\omega) \right. \\ \left. - 8m_B \ln(-\Delta_\omega) g_+^B(\omega) + 8 \frac{(P^2 - m_B \omega)}{\Delta_\omega} G_\pm^B(\omega) \right], \quad (4.79)$$

$$\mathcal{F}_4^{(d)\text{OPE}}(P^2, q^2) = \frac{f_B m_B^2}{16\pi^2} \int_0^\infty d\omega \left[-\ln(-\Delta_\omega) \Phi_\pm^B(\omega) + \frac{8}{\Delta_\omega} G_\pm^B(\omega) \right], \quad (4.80)$$

where $\Delta_\omega \equiv (P - \omega v)^2$. These are the two invariant amplitudes that enter the final LCSR relations in Eqs. (4.66) and (4.67).

The next step is to evaluate the imaginary parts of the invariant amplitudes in the variable $s = P^2$. In the expressions obtained above, the imaginary contributions originate from two different structures: the logarithmic terms $\ln(-\Delta_\omega)$ and the pole terms $1/\Delta_\omega$. We use the standard relation

$$\frac{1}{\pi} \text{Im} \ln(-\Delta_\omega) = -\theta(\Delta_\omega), \quad \frac{1}{\pi} \text{Im} \frac{1}{\Delta_\omega + i0} = -\delta(\Delta_\omega). \quad (4.81)$$

The step function generated by the logarithmic terms constrains the integration region in the variable ω through the condition $\Delta_\omega > 0$. Solving the equation $(P - \omega v)^2 = 0$ for ω , one obtains the physical root¹¹

$$\omega_- = \frac{m_B^2 + s - q^2 - \sqrt{(m_B^2 + s - q^2)^2 - 4m_B^2 s}}{2m_B}. \quad (4.82)$$

Accordingly, the step function restricts the integration region to $0 \leq \omega \leq \omega_-$, while the delta-function contribution is evaluated as

$$\delta(\Delta_\omega) = \frac{m_B}{\sqrt{(m_B^2 + s - q^2)^2 - 4m_B^2 s}} \delta(\omega - \omega_-). \quad (4.83)$$

¹¹The second solution of the equation $\Delta_\omega = 0$ corresponds to $\omega_+ = \frac{m_B^2 + s - q^2 + \sqrt{(m_B^2 + s - q^2)^2 - 4m_B^2 s}}{2m_B}$, which lies outside the physical integration region relevant for the light-cone expansion. In particular, for the kinematics considered in the LCSR derivation one has $\omega_+ > m_B$, whereas the light-cone DAs are supported in the region $\omega = \mathcal{O}(\Lambda_{\text{QCD}}) \ll m_B$. Therefore only the root ω_- contributes.

This relation follows from the standard identity

$$\delta(f(\omega)) = \sum_i \frac{\delta(\omega - \omega_i)}{|f'(\omega_i)|},$$

where ω_i are the roots of the equation $f(\omega) = \Delta_\omega = 0$. Evaluating the derivative

$$f'(\omega) = 2\omega - \frac{m_B^2 + s - q^2}{m_B}$$

at the physical root ω_- yields

$$|f'(\omega_-)| = \frac{\sqrt{(m_B^2 + s - q^2)^2 - 4m_B^2 s}}{m_B},$$

which directly leads to [Eq. \(4.83\)](#).

Using [Eqs. \(4.81\) to \(4.83\)](#), we obtain the spectral densities corresponding to the nonvanishing invariant amplitudes in [Eqs. \(4.79\) and \(4.80\)](#):

$$\begin{aligned} \text{Im } \mathcal{F}_1^{(d)\text{OPE}}(s, q^2) = & \frac{f_B m_B}{16\pi} \left[\int_0^{\omega_-} d\omega \left\{ -\Delta_\omega(s) \phi_+^B(\omega) + \left(\frac{s}{m_B} - \omega \right) \Phi_\pm^B(\omega) + 8g_+^B(\omega) \right\} \right. \\ & \left. - 8 \frac{m_B}{\sqrt{(m_B^2 + s - q^2)^2 - 4m_B^2 s}} \left(\frac{s}{m_B} - \omega_- \right) G_\pm^B(\omega_-) \right], \end{aligned} \quad (4.84)$$

$$\text{Im } \mathcal{F}_4^{(d)\text{OPE}}(s, q^2) = \frac{f_B m_B^2}{16\pi} \left[\int_0^{\omega_-} d\omega \Phi_\pm^B(\omega) - 8 \frac{m_B}{\sqrt{(m_B^2 + s - q^2)^2 - 4m_B^2 s}} G_\pm^B(\omega_-) \right]. \quad (4.85)$$

Here $\Delta_\omega(s)$ denotes Δ_ω after the replacement $P^2 \rightarrow s$.

4.3.1 Numerical Analysis

Having derived the LCSR expressions for the $B \rightarrow p$ transition form factors, we now turn to their numerical evaluation. We first specify the input parameters entering the sum rules.

In [Table 4.3](#) we collect the values and intervals of all input parameters used in the numerical analysis. Apart from the hadron masses taken from [\[96\]](#), the LCSR calculation requires the proton decay constant λ_p defined in [Eq. \(4.49\)](#). For this parameter we adopt the interval obtained in [\[112\]](#) from a two-point QCD sum-rule analysis using the same Ioffe interpolating current. The definitions and models of the B -meson DAs entering the LCSRs are summarized in [Section D.3](#). The complete set of two-particle and three-particle DAs depends on three nonperturbative input parameters. The first one is the B -meson decay constant, which is accurately determined from lattice QCD. We use the $N_f = 2 + 1 + 1$ FLAG average quoted in [\[71\]](#). Considerably less precise is the inverse moment of the twist-two DA, λ_B ,

Parameters	values/intervals	Source
Hadron masses	$m_B = 5.279 \text{ GeV}$ $m_p = 938.272 \text{ MeV}$ $m_{\Lambda_b} = 5.620 \text{ GeV}$ $m_{\pi^+} = 139.57 \text{ MeV}$	[96]
Nucleon decay constant	$\lambda_p = -(27 \pm 9) \times 10^{-3} \text{ GeV}^2$	[112]
B -meson decay constant	$f_B = 190.0 \pm 1.3 \text{ MeV}$	[71]
Inverse moment of tw2 B DA	$\lambda_B = 460 \pm 110 \text{ MeV}$	[44]
The ratio (D.25)	$R = \lambda_E^2/\lambda_H^2 = 0.30^{+0.17}_{-0.12}$	see Section D.3.1
Borel parameter squared	$M^2 = 1.5 - 2.0 \text{ GeV}^2$	[109]
Duality threshold	$s_0 = 2.25 \text{ GeV}^2$	
b -quark $\overline{\text{MS}}$ mass	$\overline{m}_b(\mu = 3 \text{ GeV}) = 4.47^{+0.04}_{-0.03} \text{ GeV}$	[96]
B^+ -meson lifetime	$\tau_B = 1.638 \pm 0.004 \text{ ps}$	[96]

Table 4.3: The input parameters used in the LCSR numerical analysis.

defined in Eq. (D.24). For this quantity we adopt the QCD sum-rule determination from [44]; see also [113]. The third parameter is the ratio $R = \lambda_E^2/\lambda_H^2$ introduced in Eq. (D.25). The interval quoted in Table 4.3 is obtained by averaging the available QCD sum-rule determinations of λ_E and λ_H from [114–116], which are based on different two-point correlation functions.

Adopting the values in Table 4.3, we implicitly assume a common renormalisation scale $\mu \sim 1 \text{ GeV}$ for λ_B and the other scale-dependent quantities entering the DAs. This scale lies in the same range as the typical Borel scale employed in the sum rules. A more refined treatment would require the calculation of radiative gluon corrections to the correlation function, which remains beyond the scope of the present analysis.

The choice of the Borel parameter M^2 and the effective continuum threshold s_0 listed in Table 4.3 follows the same strategy adopted in previous LCSR analyses of nucleon electromagnetic form factors [109, 110]. Since in the present work we employ the simplified version of the hadronic representation described above (see Eqs. (4.66) and (4.67) and the corresponding discussion), where the excited baryonic resonances are absorbed into the continuum contribution through quark–hadron duality, it is natural to use the same effective threshold $s_0 \simeq (1.5 \text{ GeV})^2 = 2.25 \text{ GeV}^2$, together with the same Borel window $M^2 = 1.5 - 2.0 \text{ GeV}^2$. Finally, for the estimate of the inclusive width $B \rightarrow X_N \psi$ used later, we employ the $\overline{\text{MS}}$ b -quark mass at the scale $\mu = 3 \text{ GeV}$ from [96].

Having specified all input parameters, we first assess the stability of the obtained form factors with respect to the Borel parameter M^2 . In Figure 4.6 we show the dependence of the form factors on the Borel mass squared. One observes a good stability behaviour throughout the adopted Borel window $M^2 = 1.5 - 2.0 \text{ GeV}^2$. In fact, the sum rules remain reasonably stable even beyond this interval, indicating a mild sensitivity of the results to the precise choice of the Borel parameter.

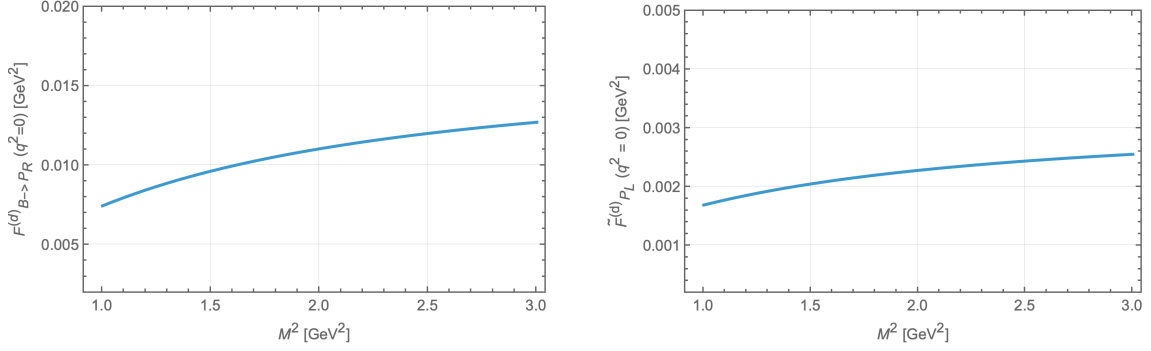


Figure 4.6: Borel-mass dependence of the form factors $F_{B \rightarrow P_R}^{(d)}$ (left panel) and $\tilde{F}_{B \rightarrow P_L}^{(d)}$ (right panel) at $q^2 = 0$, evaluated at the central values of the input parameters. The interval $M^2 = 1.5 - 2.0 \text{ GeV}^2$ is used in LCSRs.

In addition, by assessing the relative size of the proton-pole contribution and the continuum contribution above the effective threshold s_0 within the adopted Borel window, we find that the simplified hadronic ansatz employed in the present analysis leads to a numerically subdominant pole contribution in the LCSRs. This behaviour is a direct consequence of using an effective threshold that absorbs the nucleon resonances into the continuum through the duality approximation. As a result, the extracted form factors show a substantial sensitivity to the quark–hadron duality assumption underlying the continuum subtraction procedure.

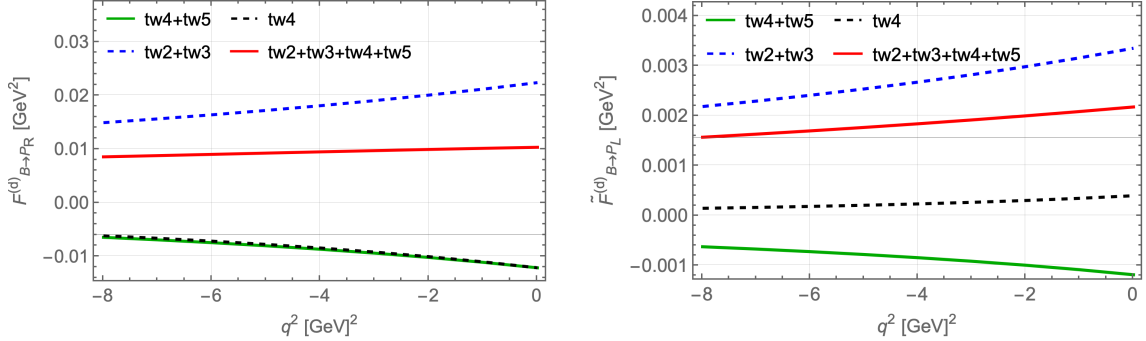


Figure 4.7: The form factors $F_{B \rightarrow P_R}^{(d)}$ (left panel) and $\tilde{F}_{B \rightarrow P_L}^{(d)}$ (right panel) in the spacelike region and at central input, shown with the red solid lines. The blue dashed (green solid) lines represent the sum of the twist-2 and twist-3 (twist-4 and twist-5) contributions, and the black dashed line represents the twist-4 contribution.

We now evaluate the form factors at their central input values in the region of validity of the LCSR, corresponding to spacelike momentum transfer $q^2 < 0$. The resulting q^2 dependence of $F_{B \rightarrow P_R}^{(d)}$ and $\tilde{F}_{B \rightarrow P_L}^{(d)}$ is displayed in Figure 4.7. For both form factors, the dominant contribution originates from the combined twist-two and twist-three terms.¹² The higher-twist contributions partially cancel the lower-twist

¹²Since the twist-three DA ϕ_-^B enters the LCSRs only through the integrated combination $\Phi_{\pm}^B$, involving also the twist-two DA ϕ_+^B , it is natural to combine the twist-two and twist-three contributions together. Analogously, the twist-four and twist-five terms appear through the combination $G_{\pm}^B$, and therefore their contributions are also shown jointly.

result and reduce the final values of the form factors. In the case of $F_{B \rightarrow P_R}^{(d)}$, the twist-four contribution already provides the dominant part of the higher-twist correction, indicating that the genuine twist-five effects remain relatively small. In contrast, for $\tilde{F}_{B \rightarrow P_L}^{(d)}$ the difference between the isolated twist-four term and the combined twist-four plus twist-five contribution is substantial, showing that the twist-five DA contribution is numerically important in this case. Overall, the hierarchy between the different twist contributions remains stable throughout the spacelike region considered.

We now turn to the discussion of the theoretical uncertainties of the obtained LCSR predictions. In the numerical analysis, the individual uncertainties associated with the input parameters listed in Table 4.3 are combined in quadrature. As representative results, we present in Table 4.4 the values of the $B \rightarrow p$ transition form factors at $q^2 = 0$ (denoted LCSR- B). The final predictions, including all two-particle DA contributions up to twist five, are shown in the column denoted by (tw2–tw5), whereas the column (tw2–tw3) is displayed only for illustration of the impact of the higher-twist corrections.

Form factor	LCSR- B (this work)		LCSR- N	
	(tw2–tw3)	(tw2–tw5)	(tw3) [95]	(tw3–tw6) [106]
$F_{B \rightarrow p_R}^{(d)}(0)$	[1.4, 3.9]	[0.4, 2.2]	[2.4, 2.8]	[0.9, 3.5]
$\tilde{F}_{B \rightarrow p_L}^{(d)}(0)$	[0.24, 0.52]	[0.14, 0.35]	0	[0.4, 0.7]
$F_{B \rightarrow p_R}^{(b)}(0)$	[2.8, 7.7]	[0.8, 4.4]	[−0.9, −0.6]	[−5.9, −2.2]
$\tilde{F}_{B \rightarrow p_L}^{(b)}(0)$	[0.5, 1.1]	[0.28, 0.70]	[0.07, 0.10]	[−0.9, −0.4]

Table 4.4: Intervals of the $B \rightarrow p$ transition form factors at $q^2 = 0$ (in units of 10^{-2} GeV²) for models (d) and (b), obtained from LCSRs with B -meson DAs (denoted as LCSR- B), are compared with the results from LCSRs with nucleon DAs (LCSR- N). Here the notation such as (tw2–tw5) indicates that contributions from twist-2 up to twist-5 are included.

The resulting uncertainties are rather sizeable. The dominant source of the error budget originates from the broad allowed intervals of the two most important nonperturbative input parameters: the inverse moment λ_B of the leading-twist B -meson DA and the proton decay constant λ_p . To quantify their impact, we separately vary λ_B and λ_p while keeping all remaining input parameters fixed at their central values. The resulting variations of the form factors at $q^2 = 0$ are collected in Table 4.5. We find that the uncertainties induced by λ_B and λ_p indeed account for a substantial fraction of the total intervals quoted in Table 4.4. At the same time, both form factors in Eqs. (4.66) and (4.67) depend on the proton decay constant through the same overall normalisation factor $1/\lambda_p$. Consequently, their ratio is independent of this parameter.

We also observe that the inclusion of the higher-twist contributions reduces the overall uncertainty of the final predictions. This effect originates from partial cancellations between the leading-twist and higher-twist terms, which simultaneously decrease the central values of the form factors and their sensitivity to the input parameters, in agreement with the behaviour illustrated in Figure 4.7.

Form factor	Uncertainty from λ_B		Uncertainty from λ_p	
	(tw2-tw3)	(tw2-tw5)	(tw2-tw3)	(tw2-tw5)
$F_{B \rightarrow p_R}^{(d)}(0)$	[1.6, 3.5]	[0.5, 2.0]	[1.7, 3.4]	[0.8, 1.6]
$\tilde{F}_{B \rightarrow p_L}^{(d)}(0)$	[0.29, 0.39]	[0.16, 0.29]	[0.25, 0.49]	[0.16, 0.33]
$F_{B \rightarrow p_R}^{(b)}(0)$	[3.2, 6.9]	[1.0, 4.1]	[3.3, 6.7]	[1.6, 3.1]
$\tilde{F}_{B \rightarrow p_L}^{(b)}(0)$	[0.59, 0.77]	[0.32, 0.59]	[0.5, 1.0]	[0.33, 0.65]

Table 4.5: The same form factors from LCSRs with B -meson DAs as in Table 4.4. Shown are the intervals stemming from separate variations of the parameter λ_B or λ_p with all other input parameters fixed at their central values.

Finally, it is important to stress that the quoted uncertainties should not be interpreted statistically in the Gaussian sense. In particular, both λ_B and λ_p are themselves extracted from QCD sum rules, where the quoted intervals reflect systematic variations of auxiliary sum-rule parameters rather than probability distributions. For this reason, we prefer to present our final LCSR predictions in the form of intervals instead of central values with Gaussian errors.

In Table 4.4 we also quote previous predictions for the $B \rightarrow p$ form factors obtained from LCSRs based on nucleon DAs, denoted by LCSR- N . The original analysis of [95], performed at twist-three accuracy, was later extended in [106] to include contributions up to twist six. Comparing the LCSR- B and LCSR- N approaches, one immediately notices that the simple relations between the form factors in models (d) and (b) derived in Eq. (4.72) are absent in the LCSR- N framework at any twist order. In addition, within the LCSR- N approach the form factors in model (b) acquire opposite signs relative to those in model (d).

At the same time, it should be emphasized that the notion of twist for nucleon DAs differs fundamentally from the twist classification used for B -meson DAs. Therefore, a direct term-by-term comparison between twist contributions in the two approaches is not meaningful. Nevertheless, the overall hierarchy of the twist expansion remains an important criterion for the reliability of the OPE. In the present LCSR- B analysis this hierarchy is reasonably well respected, as illustrated in Figure 4.7. In contrast, the LCSR- N results of [106] exhibit very large higher-twist effects.

To investigate this point in more detail, we independently evaluated the sum-rule expressions of [106] in the spacelike region using the same input parameters adopted there. In particular, focusing on model (b) at $q^2 = -4 \text{ GeV}^2$, where the twist expansion is expected to be reliable, we find for the form factor $F_{B \rightarrow p_R}^{(b)}$ that the twist-four contribution exceeds the twist-three term by roughly a factor of four, while the twist-five contribution amounts to about 60% of the twist-three one. For the second form factor, $\tilde{F}_{B \rightarrow p_L}^{(b)}$, the enhancement is even more pronounced, with the twist-four contribution exceeding the twist-three term by approximately one order of magnitude. This behaviour explains the substantial difference between the twist-three and twist-six predictions quoted in the first two columns of Table 4.4, especially for model (b).

For model (d) the higher-twist effects in the LCSR- N approach are considerably less dramatic, and the agreement with our LCSR- B results is correspondingly

better. Moreover, once the rather large uncertainties of both methods are taken into account, the predicted intervals still overlap marginally. Nevertheless, the convergence pattern of the OPE in the LCSR- N framework, particularly for model (b), appears questionable and deserves further investigation.

The LCSR predictions derived above are valid only in the spacelike region of momentum transfer $q^2 < 0$, where the OPE is expected to converge reliably. In contrast, the physical decay region for the process $B \rightarrow p\psi$ corresponds to positive timelike momentum transfer, $0 \leq q^2 \leq (m_B - m_p)^2$. Therefore, in order to obtain phenomenological predictions for the physical decay amplitudes and widths, the form factors calculated from LCSRs in the spacelike domain must be extrapolated analytically to the physical timelike region.

To extrapolate the LCSR predictions to the physical timelike region, we employ the standard z -expansion method widely used for heavy-to-light form factors. We introduce the conformal variable

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}, \quad (4.86)$$

with $t_+ = (m_{\Lambda_b} + 2m_\pi)^2$. For simplicity, we choose the second free parameter of the mapping to be $t_0 = 0$, such that the point $q^2 = 0$ corresponds to $z = 0$.

At this stage it is useful to recall that the correlation function in Eq. (4.44) depends on two independent kinematic variables, P^2 and q^2 , which correspond to two different hadronic channels. The variable P^2 is associated with the baryonic channel interpolated by the Ioffe current and is therefore saturated by hadronic states carrying the proton quantum numbers. This is the channel in which the dispersion relation and Borel transformation were constructed in the derivation of the LCSRs. In contrast, the variable q^2 corresponds to the channel generated by the local three-quark operators $\overline{\mathcal{O}}_{(X)}$ in Eqs. (4.40) and (4.41). Consequently, its analytic structure is determined by hadronic states carrying the same quantum numbers as these operators. The lowest singularity in this channel is given by the Λ_b pole, whereas the continuum contribution starts from the $\Lambda_b\pi\pi$ threshold and receives additional contributions from higher excited and multiparticle states carrying the same quantum numbers. This is precisely the reason for choosing $t_+ = (m_{\Lambda_b} + 2m_\pi)^2$ in the conformal mapping variable defined in Eq. (4.86).¹³

Under this transformation, the spacelike region where the LCSR is evaluated and the physical decay region are mapped to

$$-8.0 \text{ GeV}^2 < q^2 < 0 \quad \rightarrow \quad 0.083 \geq z \geq 0, \quad (4.87)$$

$$m_p^2 < q^2 < (m_B - m_p)^2 \quad \rightarrow \quad -0.067 > z > -0.20. \quad (4.88)$$

Choosing the lowest physical threshold in the definition of t_+ ensures that only the Λ_b pole remains inside the unit disk $|z| < 1$ to which the full complex q^2 plane is mapped. To remove this pole contribution, we multiply the form factors by the factor $(1 - q^2/m_{\Lambda_b}^2)$. The resulting functions become analytic inside the unit disk

¹³In [95, 106], a similar z -expansion was employed assuming instead the threshold $t_+ = (m_B + m_p)^2$. Here we use the physically lowest threshold corresponding to the $\Lambda_b\pi\pi$ state. We verified that using the higher threshold adopted in those works changes the extrapolated form factors by less than 1%.

and can therefore be expanded around $z = 0$. Since both the spacelike region used in the LCSR and the physical decay region correspond to rather small values of $|z|$, the expansion converges rapidly and can be truncated after the first few terms.

More specifically, we employ the standard Bourrely–Caprini–Lellouch (BCL) parametrisation [117] truncated at $\mathcal{O}(z^2)$:

$$F_{B \rightarrow p_R}^{(d)}(q^2) = \frac{F_{B \rightarrow p_R}^{(d)}(0)}{1 - q^2/m_{\Lambda_b}^2} \left\{ 1 + \beta_{B \rightarrow p_R}^{(d)} \left[z(q^2) - z(0) + \frac{1}{2} (z(q^2)^2 - z(0)^2) \right] \right\}. \quad (4.89)$$

This parametrisation is specified by two quantities: the form-factor normalisation at $q^2 = 0$ and the slope parameter $\beta_{B \rightarrow p_R}^{(d)}$. An analogous expression with the slope parameter $\tilde{\beta}_{B \rightarrow p_L}^{(d)}$ is used for the form factor $\tilde{F}_{B \rightarrow p_L}^{(d)}(q^2)$.

For the model-(b) form factors we employ the relations in Eq. (4.72), which imply

$$F_{B \rightarrow p_R}^{(b)}(0) = 2F_{B \rightarrow p_R}^{(d)}(0), \quad \tilde{F}_{B \rightarrow p_L}^{(b)}(0) = 2\tilde{F}_{B \rightarrow p_L}^{(d)}(0), \quad (4.90)$$

together with equal slope parameters:

$$\beta_{B \rightarrow p_R}^{(b)} = \beta_{B \rightarrow p_R}^{(d)}, \quad \tilde{\beta}_{B \rightarrow p_L}^{(b)} = \tilde{\beta}_{B \rightarrow p_L}^{(d)}. \quad (4.91)$$

We then fit the parametrisation in Eq. (4.89) to the LCSR results obtained for the form factors in the spacelike region $q^2 \leq 0$. We first employ a simplified fitting procedure in which the normalisation of the form factor at $q^2 = 0$ is fixed to the LCSR prediction, while only the slope parameter is fitted. More specifically, for the form factor $F_{B \rightarrow p_R}^{(d)}(q^2)$ we determine the parameter $\beta_{B \rightarrow p_R}^{(d)}$ using the LCSR results evaluated at eight equidistant points in the interval defined in Eq. (4.87). The same procedure is repeated for the form factor $\tilde{F}_{B \rightarrow p_L}^{(d)}(q^2)$. To estimate the uncertainty of the fitted slope parameters, the input value of the form factor at $q^2 = 0$ is varied within the parametric uncertainty obtained from the LCSR analysis. The induced variation is then propagated to the fitted slope parameters, yielding asymmetric uncertainties for $\beta_{B \rightarrow p_R}^{(d)}$ and $\tilde{\beta}_{B \rightarrow p_L}^{(d)}$. The resulting parameters of the z -expansion are collected in Table 4.6. For the model-(b) form factors, the relations in Eqs. (4.90) and (4.91) are used.

$F_{B \rightarrow p_R}^{(d)}(0)$	$1.0^{+1.2}_{-0.6}$	$\beta_{B \rightarrow p_R}^{(d)}$	$0.99^{+3.4}_{-1.4}$
$\tilde{F}_{B \rightarrow p_L}^{(d)}(0)$	$0.22^{+0.13}_{-0.08}$	$\tilde{\beta}_{B \rightarrow p_L}^{(d)}$	$-1.78^{+0.76}_{-0.17}$
$F_{B \rightarrow p_R}^{(b)}(0)$	$2.0^{+2.4}_{-1.2}$	$\beta_{B \rightarrow p_R}^{(b)}$	$0.99^{+3.4}_{-1.4}$
$\tilde{F}_{B \rightarrow p_L}^{(b)}(0)$	$0.44^{+0.26}_{-0.16}$	$\tilde{\beta}_{B \rightarrow p_L}^{(b)}$	$-1.78^{+0.76}_{-0.17}$

Table 4.6: Parameters of the z -expansion in Eq. (4.89) obtained from the fit of LCSRs. Presented are the $B \rightarrow p$ form factors at $q^2 = 0$ (in units of 10^{-2} GeV^2) and the dimensionless slope parameters.

The advantage of the fixed-normalisation fit is its simplicity and its direct correspondence to the uncertainty intervals obtained from the LCSR analysis. However,

this procedure does not explicitly account for possible correlations between the normalisation and slope parameters entering the z -expansion. To estimate the size of such effects, we also carried out an alternative fit in which the normalisation $F_{B \rightarrow p_R}^{(d)}(0)$ is allowed to vary together with the slope parameter. In this case, the covariance matrix derived from the Hessian automatically incorporates the correlations between both quantities. We find that the resulting predictions for the branching fractions remain compatible, within uncertainties, with those obtained in the fixed-normalisation approach. This indicates that the effect of parameter correlations is numerically insignificant.

Having extrapolated the form factors to the physical timelike region, we are now in a position to predict the branching fraction for the exclusive decay $B^+ \rightarrow p\Psi$. Using the decomposition in Eq. (4.43), the decay amplitude in Eq. (4.42) can be written as

$$\begin{aligned} \mathcal{A}^{(X)}(B^+ \rightarrow p\Psi) = G_{(X)} & \left[F_{B \rightarrow p_R}^{(X)}(q^2) \bar{u}_{pR}(P) + F_{B \rightarrow p_L}^{(X)}(q^2) \bar{u}_{pL}(P) \right. \\ & \left. + \tilde{F}_{B \rightarrow p_R}^{(X)}(q^2) \bar{u}_{pR}(P) \frac{\not{q}}{m_p} + \tilde{F}_{B \rightarrow p_L}^{(X)}(q^2) \bar{u}_{pL}(P) \frac{\not{q}}{m_p} \right] u_\Psi^c(q), \end{aligned} \quad (4.92)$$

evaluated at $q^2 = m_\Psi^2$. The total decay width is obtained in the standard way from the spin-summed squared amplitude integrated over the two-body phase space:

$$\Gamma(B^+ \rightarrow p\Psi)_{(X)} = \frac{1}{2m_B} \int d\Phi_2 \overline{|\mathcal{A}^{(X)}|^2}, \quad (4.93)$$

where

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(p_B - p_p - p_\Psi) \frac{d^3 p_p}{(2\pi)^3 2E_p} \frac{d^3 p_\Psi}{(2\pi)^3 2E_\Psi} \quad (4.94)$$

denotes the Lorentz-invariant two-body phase-space measure. In the present case, the momenta are identified as $p_B = P + q$, $p_p = P$, and $p_\Psi = q$. For a two-body decay, the phase-space integration can be carried out analytically and yields the familiar kinematic factor $\lambda^{1/2}(m_B^2, m_p^2, m_\Psi^2)$, where $\lambda(x, y, z)$ is the Källén function defined in Eq. (B.11). Retaining only the nonvanishing form factors, the decay width for real-valued form factors reads

$$\begin{aligned} \Gamma_{(X)}(B^+ \rightarrow p\Psi) = \frac{|G_{(X)}|^2 \lambda^{1/2}(m_B^2, m_p^2, m_\Psi^2)}{16\pi m_B^3} & \left\{ \left[\left(F_{B \rightarrow p_R}^{(X)}(m_\Psi^2) \right)^2 \right. \right. \\ & + \frac{m_\Psi^2}{m_p^2} \left(\tilde{F}_{B \rightarrow p_L}^{(X)}(m_\Psi^2) \right)^2 \left. \right] (m_B^2 - m_p^2 - m_\Psi^2) \\ & + 2m_\Psi^2 F_{B \rightarrow p_R}^{(X)}(m_\Psi^2) \tilde{F}_{B \rightarrow p_L}^{(X)}(m_\Psi^2) \left. \right\}. \end{aligned} \quad (4.95)$$

The branching fraction is then obtained from

$$\text{BR}_{(X)}(B^+ \rightarrow p\Psi) = \tau_{B^+} \Gamma_{(X)}(B^+ \rightarrow p\Psi). \quad (4.96)$$

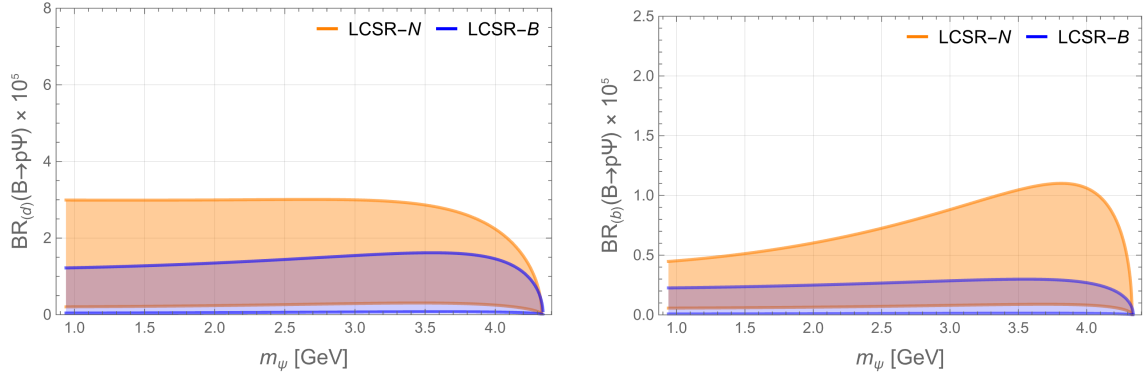


Figure 4.8: Branching fraction predicted for the $B^+ \rightarrow p\Psi$ decay in model (d) (left panel) and model (b) (right panel). The blue bands represent the results of this work obtained with the LCSR- B method. For comparison, the orange bands show the results obtained using the method with nucleon DAs (LCSR- N).

The resulting branching fractions are shown in [Figure 4.8](#) as functions of the dark-fermion mass m_Ψ .

The uncertainty bands are obtained by varying independently the z -expansion parameters entering the decay rate in [Eq. \(4.95\)](#). For each form factor, both the normalisation and the slope parameter are varied over the ranges given in [Table 4.6](#). We use three representative values for each parameter, corresponding to its central, maximal, and minimal values. For every fixed value of m_Ψ , the branching fraction is then evaluated over the full set of parameter combinations, and the resulting uncertainty band is defined by the global minimum and maximum obtained from this scan. This procedure provides a conservative estimate of the theoretical uncertainty, since it explores the full allowed parameter ranges without assuming any particular statistical distribution or correlation pattern among the fit parameters. Consequently, the resulting bands are expected to encompass the maximal variation of the branching-fraction predictions compatible with the adopted LCSR input.

Moreover, in [Figure 4.8](#) we compare our predictions with the results obtained in the LCSR- N approach of [\[106\]](#). In both approaches the effective couplings in [Eq. \(4.39\)](#) are fixed to their maximal allowed values (see [Table D.3](#)),

$$|G_{(d)}|^2 = 1.0 \times 10^{-13} \text{ GeV}^{-4}, \quad |G_{(b)}|^2 = 4.7 \times 10^{-15} \text{ GeV}^{-4},$$

which correspond to the upper bounds inferred from the ATLAS and CMS searches for heavy coloured particles, as discussed in [\[31\]](#).¹⁴ We find that the predictions obtained from the two LCSR methods are compatible within the rather large theoretical uncertainties. Nevertheless, throughout the full kinematic range our LCSR- B results tend to lie below the corresponding LCSR- N predictions. Moreover, the uncertainty bands obtained in the present analysis are noticeably narrower.

Now that we have obtained the final predictions for the form factors and the corresponding branching fractions, it is worthwhile to perform a more detailed comparison between the two complementary approaches based on B -meson DAs

¹⁴In [\[95, 106\]](#), equal values $|G_{(d)}|^2 = |G_{(b)}|^2 = 10^{-13} \text{ GeV}^{-4}$ were used.

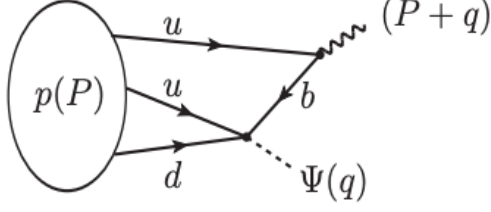


Figure 4.9: Representative diagram entering the LCSR- N approach based on proton DAs. In this case only the b -quark field is contracted perturbatively, while the three light-quark fields are absorbed into the proton DAs.

(LCSR- B) and nucleon DAs (LCSR- N). Such a comparison is particularly instructive since both methods rely on different realisations of the light-cone expansion and organize the nonperturbative dynamics in substantially different ways.

- **Structure of the correlation function and OPE diagrams:** In the LCSR- B framework, the nonperturbative B -meson matrix element contains only two light-cone partons, such that two quark fields in the correlation function are contracted perturbatively, generating a closed quark loop in the OPE diagrams (see Figure 4.5). In contrast, the LCSR- N approach is based on nucleon DAs involving three quark fields, so that only the b -quark propagator is contracted perturbatively (see Figure 4.9). As a result, the analytical structure of the resulting form factors becomes more complicated in the LCSR- N approach, in particular when extracting the imaginary parts required for the dispersion relations.
- **Structure of the light-cone expansion:** Both approaches are based on a twist expansion on the light cone, but the organisation and implementation of higher-twist effects differ substantially. In the LCSR- B framework, higher-twist two-particle B -meson DAs can be incorporated rather directly through the vacuum-to- B matrix elements, as done in the present work up to twist five. In contrast, extending the LCSR- N approach to higher twists is technically more involved, since it requires the full set of higher-twist nucleon DAs together with their corresponding contributions to the correlation function. This leads to a much more complicated analytical structure of the resulting sum rules.
At the same time, the concept of twist for B -meson DAs is not identical to the twist classification employed for nucleon DAs. Therefore, a direct term-by-term comparison of individual twist contributions between the two approaches is not meaningful.
- **Interpolating currents and hadronic representation:** An important conceptual difference between the two approaches concerns the choice of interpolating current and the resulting structure of the hadronic representation. In the present LCSR- B framework, the proton is interpolated by a baryonic current, which leads to a considerably more involved hadronic spectrum in the proton channel. In particular, besides the proton pole itself, one encounters

negative-parity baryon states and, in the intermediate region, the broad Roper resonance, which cannot be represented by a simple narrow-pole contribution. A fully realistic treatment of this hadronic structure would require a multi-resonance parametrisation together with additional hadronic inputs, such as the decay constants of the excited baryons. In particular, no reliable determination of the Roper-resonance decay constant is currently available. For this reason, we adopted a simplified hadronic ansatz in which only the proton pole is separated explicitly, while all excited and continuum states are absorbed into the duality continuum. As a consequence, the resulting sum rules exhibit an enhanced sensitivity to the continuum contribution and rely more strongly on the quark–hadron duality approximation.

In contrast, the LCSR- N approach employs an interpolating current for the B meson. The hadronic structure in this channel is much simpler, since no nearby opposite-parity resonance analogous to the nucleon case appears below the continuum threshold. Consequently, the ground-state pole remains parametrically more dominant and the continuum contribution is relatively more suppressed compared to the LCSR- B framework.

- **Three-particle contributions:** In the LCSR- B framework, the contributions of the three-particle B -meson DAs vanish identically for both models (d) and (b). For model (d), the cancellation occurs because the soft-gluon emission from the d -quark propagator is equal in magnitude and opposite in sign to the corresponding u -quark emission contribution, as shown in Eq. (4.77). In contrast, for model (b) the three-particle DA contributions vanish term by term already at the level of the individual diagrams, see Eq. (4.78). These cancellations originate from the interplay between the Dirac structure of the effective operators and the specific linear combination defining the Ioffe current. In addition, the massless limit adopted for the light quarks also plays a role in these cancellations; retaining finite light-quark masses would generate nonvanishing three-particle contributions proportional to the light-quark masses. No analogous cancellation mechanism is present in the LCSR- N approach.
- **Relations between models (d) and (b):** A characteristic feature of the present LCSR- B calculation is the simple relation in Eq. (4.72). These relations hold for the complete two-particle DA contributions. The three-particle DA contributions, however, do not satisfy such relations individually. In the LCSR- N approach, no analogous relations between models (d) and (b) emerge at any twist order, and the model- (b) form factors even acquire opposite signs relative to those in model (d) (see Table 4.4).
- **Numerical size of the form factors:** Although the predictions of both methods overlap within uncertainties, the form factors obtained in the present LCSR- B analysis are systematically smaller in magnitude than the corresponding LCSR- N results. This difference propagates directly to the predicted branching fractions.
- **Uncertainty pattern:** The dominant uncertainties in the present work originate from the inverse moment λ_B and the proton decay constant λ_p . Even

with these sizeable uncertainties, the resulting bands remain narrower than those obtained in the LCSR- N approach. Moreover, in the present analysis the inclusion of higher-twist contributions tends to stabilize the final predictions due to partial cancellations between different twist components.

- **Hierarchy of twist contributions and convergence of the OPE:** The overall hierarchy of the twist expansion provides an important criterion for assessing the convergence and reliability of the OPE in both sum-rule approaches. In the present LCSR- B analysis, the lowest-twist contributions dominate the form factors, while the higher-twist corrections remain systematically suppressed throughout the spacelike region considered. This hierarchical pattern, together with the observed stability under variation of the Borel parameter, indicates a reasonably controlled OPE.

In contrast, the situation is substantially different in the LCSR- N framework, especially for model (b). Using the sum-rule expressions and input parameters of Ref. [106], we evaluated the separate twist contributions at $q^2 = -4 \text{ GeV}^2$. For the form factor $F_{B \rightarrow p_R}^{(b)}$, the twist-four contribution is approximately four times larger than the twist-three term, while the twist-five correction still amounts to roughly 60% of the twist-three contribution. An even stronger enhancement is observed for $\tilde{F}_{B \rightarrow p_L}^{(b)}$, where the twist-four contribution exceeds the twist-three one by nearly an order of magnitude.

Tracing the origin of this enhancement in more detail, one finds that the dominant effect arises from the twist-four $\mathcal{T}_{2,4}$ -contributions in the decomposition of the nucleon DAs [106]. These terms introduce the parameter f_2^d , directly related to the next-to-conformal coefficient ξ_{10} , whose numerical value is known rather poorly and is typically assigned a conservative uncertainty of about 50% [118]. As a result, the twist-four correction becomes anomalously large in model (b) and significantly enhances the predicted branching fraction. In contrast, for model (d) these specific contributions cancel identically in the course of the calculation, explaining why the higher-twist effects remain much more moderate in that case.

Overall, this pattern implies that higher-twist terms dominate the final result in the LCSR- N framework for model (b), raising questions about the convergence of the corresponding OPE and indicating that the higher-twist structure deserves further investigation.

- **Overall consistency:** Despite these conceptual and numerical differences, both approaches lead to branching-fraction predictions that remain compatible within their relatively large theoretical uncertainties. This agreement provides a nontrivial cross-check of the general magnitude of the exclusive $B \rightarrow p\Psi$ transition amplitudes.

4.4 Ratio of Exclusive to Inclusive Decay Widths

The branching-fraction predictions shown in Figure 4.8 were obtained by fixing the effective couplings $G_{(d,b)}$ to their maximal values allowed by the current LHC

constraints from ATLAS and CMS searches for heavy coloured particles. On the other hand, the inclusive contribution generated by the corresponding quark-level transition $b \rightarrow \psi ud$ was calculated within the HQE framework in [Section 4.2](#). Having both the exclusive and inclusive decay rates at hand, we can now relate them within the Mesogenesis framework and derive a lower bound on the exclusive branching fraction.

Using the condition on the inclusive decay rate in [Eq. \(4.4\)](#), one can obtain a corresponding lower bound on the exclusive decay by considering the ratio of the exclusive and inclusive decay widths. Multiplying this ratio by the lower limit on the inclusive branching fraction directly yields a lower bound for the exclusive branching fraction. To this end, we introduce the exclusive-to-inclusive ratio

$$R_{[\text{excl}/\text{incl}]}(B^+ \rightarrow p\Psi) \equiv \frac{\Gamma(B^+ \rightarrow p\Psi)}{\Gamma(B^+ \rightarrow X_N\Psi)} \times 10^{-4} = \frac{\text{BR}(B^+ \rightarrow p\Psi)}{\text{BR}(B^+ \rightarrow X_N\Psi)} \times 10^{-4}. \quad (4.97)$$

Here X_N denotes all possible kinematically accessible baryonic hadronic states produced after the underlying quark-level transition $\bar{b} \rightarrow ud\Psi$. Together with the spectator u -quark of the B^+ meson, the produced quarks form a visible ud system carrying the quantum numbers of a proton. The lowest such state is the proton itself, corresponding to the exclusive channel $B^+ \rightarrow p\Psi$, whereas the inclusive state X_N also contains excited nucleon and baryonic resonance states, as well as multiparticle hadronic configurations such as N^* , Δ , $N^*\pi$, $p\pi\pi$, and similar states, whenever kinematically allowed.

For the inclusive decay width entering the ratio [Eq. \(4.97\)](#), we use the HQE prediction including all contributions up to dimension six. More specifically, we include the dimension-three free-quark contribution, the dimension-five chromomagnetic and kinetic corrections, and the dimension-six Darwin-term contribution for the two-quark operators, together with the dimension-six four-quark operator contributions. The corresponding expressions can be found in [Eqs. \(4.26\) to \(4.33\)](#).

The resulting lower limits on the exclusive branching fraction are shown in [Figure 4.10](#) for models (d) and (b). The solid green curves represent the central predictions obtained from the exclusive-to-inclusive ratio in [Eq. \(4.97\)](#), while the dashed green curves indicate the maximal values of the lower limits after including the estimated theoretical uncertainties. For comparison, the shaded orange and blue regions display the current experimental upper bounds from BaBar [\[98\]](#) and Belle/Belle II [\[97\]](#) searches for $B \rightarrow p + \text{invisible}$ decays. The remaining white regions therefore correspond to the parameter space currently allowed by experiment. As discussed in [Section 4.2](#), the inclusion of subleading terms in the HQE reveals specific regions of the dark-fermion mass m_ψ where the OPE description ceases to be reliable. More precisely, for sufficiently large values of m_ψ , the subleading power corrections become larger than the leading dimension-three contribution, signaling a breakdown of the convergence of the expansion. In model (d) this behaviour occurs for $m_\psi \sim 2$ GeV, whereas in model (b) it appears around $m_\psi \sim 3$ GeV. For this reason, when evaluating the exclusive-to-inclusive ratio and the corresponding lower bounds in [Figure 4.10](#), we restrict the plots to the regions below these values of m_ψ , where the HQE remains under theoretical control.

The uncertainty band shown in [Figure 4.10](#), enclosed between the solid and dashed green curves, is obtained following the same conservative procedure adopted

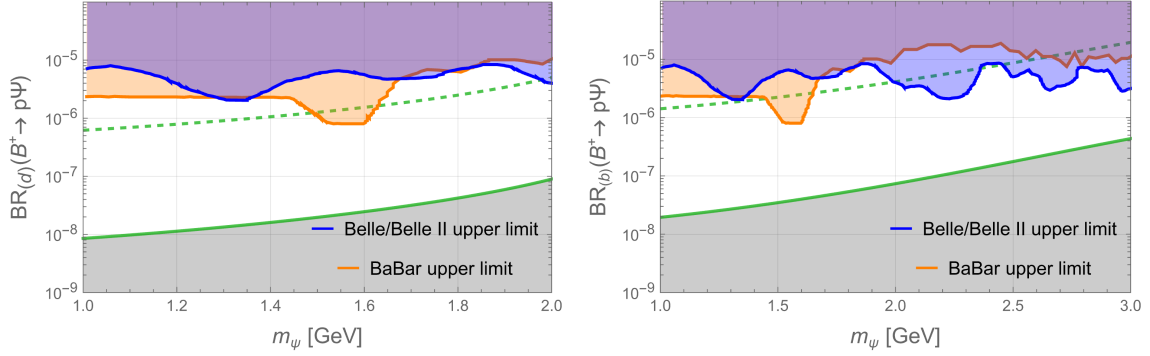


Figure 4.10: Branching fraction $\text{BR}(B^+ \rightarrow p\Psi)$ as a function of m_Ψ for model (d) (left panel) and model (b) (right panel). The solid green lines present the predicted lower limit in Eq. (4.97), and the dashed green lines indicate the maximal value of this limit within estimated uncertainties; orange and blue shaded regions display experimental upper limits set by, respectively, BaBar [98] and Belle/Belle II [97] measurements of the $B \rightarrow p + \text{invisible}$ decays. The white areas indicate the parameter space allowed by current experimental data.

for the exclusive branching fractions in Figure 4.8. More specifically, the parameters entering the exclusive-to-inclusive ratio are varied independently over their allowed ranges, and the extremal values obtained from the full scan are used to define the uncertainty interval. In addition, we assign an extra 10% uncertainty to the inclusive width in order to account for residual theoretical effects not included explicitly in the calculation, such as renormalisation-scale dependence, scheme ambiguities, and neglected higher-order perturbative and power corrections within the HQE. Numerically, the total uncertainty is dominated by the form factors $F_{B \rightarrow p_R}^{(d)}(0)$ and $F_{B \rightarrow p_R}^{(b)}(0)$ determined in Table 4.4. To remain conservative in deriving phenomenological constraints, we interpret the lower edge of the uncertainty band as our final prediction for the lower bound on $\text{BR}(B^+ \rightarrow p\Psi)$.

Important note: Shortly after completing our work on B -Mesogenesis, the study [119] appeared, employing LHC searches for jets accompanied by missing energy to constrain baryon-number violating operators. A subset of these operators corresponds to the Mesogenesis scenarios considered in the present work. Their analysis indicates that the flavour structure associated with the quark combination bud is strongly constrained by current data, whereas scenarios involving heavier flavours, such as bcs , remain phenomenologically viable. In the latter case, the corresponding exclusive decay channels are expected to involve charmed baryons, including $B^{0,+} \rightarrow \Xi_c^{0,+}\Psi$ and $B_s \rightarrow \Omega_c^0\Psi$. This observation further highlights the importance of a flavour-flexible framework for exclusive form-factor calculations. In particular, the LCSR approach developed in the present work can be generalised to a broad class of baryonic final states beyond the proton channel considered here. At the same time, our inclusive HQE analysis already incorporates the complete set of Mesogenesis operators, independently of the flavour assignment, making the framework suitable for studying a wide variety of exclusive-to-inclusive decay patterns in future investigations.

Chapter 5

Summary and Outlook

In this thesis, we have explored several aspects of heavy-hadron lifetimes within and beyond the SM, employing both the HQE and the method of LCSR. The results presented here collectively contribute to a more complete theoretical understanding of inclusive and exclusive B -meson decays and provide a model-independent framework for testing new-physics effects in heavy-flavour observables.

In the SM part of this work, we extended the theoretical description of inclusive non-leptonic B -hadron decays within the HQE by deriving previously unknown higher-order contributions. These include the complete leading-order dimension-seven two-quark Wilson coefficients for the CKM-favoured decay channels $b \rightarrow c\bar{u}d$ and $b \rightarrow c\bar{c}s$, as well as the previously missing dimension-five and dimension-six contributions of the SM QCD-penguin operators. Together, these results provide essential ingredients for future high-precision predictions of inclusive B -hadron decay widths and lifetime ratios.

In the second part of this thesis, we extended the HQE to a model-independent effective Hamiltonian describing $|\Delta B| = 1$ non-leptonic b -hadron decays beyond the SM. We derived the leading-order Wilson coefficients of the HQE up to and including the dimension-six contributions, comprising the free-quark term, the chromomagnetic correction, the two-quark Darwin contribution, and the previously uncalculated WA topology of the dimension-six four-quark operators. The resulting expressions provide a model-independent framework for studying short-distance new physics in inclusive b -hadron decays and can be readily matched onto a broad class of ultraviolet completions.

Finally, we investigated the phenomenological implications of B -Mesogenesis, in which the origin of the baryon asymmetry of the Universe is linked to the dark-matter sector through exotic b -hadron decays. We analysed the impact of the new interactions on inclusive B -meson decay widths and lifetime ratios within the HQE, including power corrections up to the Darwin term. We further studied the convergence of the HQE in the presence of a massive dark-sector fermion in the final state, identifying the region of parameter space where the operator product expansion remains reliable.

Complementing the inclusive analysis, we performed an exclusive calculation of the baryon-number-violating decay $B^+ \rightarrow p\psi$ using the method of LCSR. By computing the relevant $B \rightarrow p$ transition form factors from B -meson distribution amplitudes, we predicted the corresponding branching fraction and established a quantitative connection between the inclusive and exclusive descriptions. This allowed us to derive a lower bound on $\text{BR}(B^+ \rightarrow p\psi)$. A comparison with the experimental upper bounds from BaBar and Belle/Belle II then enabled us to determine the regions of parameter space that are currently allowed by experiment.

Several directions for future research naturally follow from the work presented in this thesis. On the theoretical side, an important next step is the computation of

next-to-leading-order (NLO) QCD corrections to the Darwin term. These corrections are crucial for further reducing the theoretical uncertainty in lifetime predictions and for enabling a more precise comparison with experimental data.

Another promising direction is the detailed matching of specific BSM scenarios—such as extended Higgs sectors, leptoquarks, or heavy neutral fermions—onto the general operator basis developed here. This would allow for direct phenomenological applications of the formalism and provide quantitative constraints on the corresponding Wilson coefficients from lifetime measurements.

Furthermore, a natural next step is the synthesis of these calculations: integrating the new dimension-seven SM corrections with the full set of dimension-six BSM operators to perform a global phenomenological analysis. Such a study would sharpen the SM prediction for lifetimes and lifetime ratios while providing a consistent framework to constrain general New Physics.

The LCSR framework can also be extended to a broader class of exclusive channels, including baryonic and multi-body decays, where hadronic uncertainties remain significant. Such analyses would improve our understanding of the transition between inclusive and exclusive regimes and could play an important role in testing quark-hadron duality.

Moreover, future experimental progress in flavour physics—at Belle II, LHCb Upgrade II, and future colliders—will provide unprecedented precision in heavy-hadron lifetime measurements. Confronting these data with the refined theoretical predictions developed in this thesis will offer a powerful probe of both SM dynamics and possible new-physics contributions.

Finally, the questions addressed in this thesis represent only a small part of a much broader programme in heavy-flavour physics. The theoretical tools developed and extended here—including effective field theories, the HQE, and LCSRs—are sufficiently versatile to be applied to a wide range of precision studies within and beyond the SM. As experimental measurements continue to improve, these methods will remain indispensable for interpreting the data and for uncovering possible signs of new physics. In this sense, this thesis is not only about heavy-hadron lifetimes, but about developing a broader framework for precision flavour physics. It is, indeed, *more than a lifetime*.

Appendix A

Collection of Useful Formulas

The S -matrix element S_{fi} connects the asymptotic initial and final states in scattering processes. It is commonly written as

$$S_{fi} = \langle f | \hat{S} | i \rangle, \quad (\text{A.1})$$

where $|i\rangle$ and $|f\rangle$ denote the initial and final states, respectively. The S -operator is the time-ordered exponential of the interaction Lagrangian $\mathcal{L}_{\text{int}}$,

$$\hat{S} = T \exp \left[i \int d^4x \mathcal{L}_{\text{int}}(x) \right], \quad (\text{A.2})$$

which can be expanded to second order as

$$S_{fi} = \langle f | 1 + i \int d^4x T\{\mathcal{L}_{\text{int}}(x)\} - \frac{1}{2} \int d^4x d^4y T\{\mathcal{L}_{\text{int}}(x)\mathcal{L}_{\text{int}}(y)\} + \cdots | i \rangle. \quad (\text{A.3})$$

Here, the first-order term gives the leading transition amplitude, while the second-order term accounts for processes with two interaction vertices or loop corrections. The transition amplitude T_{fi} is then defined through

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) T_{fi}. \quad (\text{A.4})$$

The interaction Lagrangian consists of QCD and electroweak contributions:

$$\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{EW}}. \quad (\text{A.5})$$

In this thesis, we focus on weak transitions of quarks. The relevant part of the electroweak Lagrangian for quark flavour-changing processes is

$$\mathcal{L}_{\text{weak}} = -\frac{g}{\sqrt{2}} V_{ij} [\bar{u}_i \gamma^\mu P_L d_j] W_\mu^+ + \text{h.c.}, \quad (\text{A.6})$$

where $P_L = (1 - \gamma_5)/2$, V_{ij} are CKM matrix elements, u_i and d_j denote up- and down-type quark fields, and W_μ^+ is the charged weak boson.

The QCD Lagrangian for quarks and gluons is

$$\mathcal{L}_{\text{QCD}} = \bar{q}(i\gamma^\mu D_\mu - m_q)q - \frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu}, \quad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c. \quad (\text{A.7})$$

where q denotes the quark fields, m_q their masses, $G_{\mu\nu}^a$ is the gluon field strength tensor, g_s the strong coupling, T^a the $SU(3)_c$ generators, and A_μ^a the gluon fields.

The covariant derivative acting on a field in the fundamental representation is defined by

$$iD_\mu = i\partial_\mu + g_s A_\mu^a t^a, \quad (\text{A.8})$$

where $t^a = \lambda^a/2$ are the generators of $SU(3)_c$. For fields transforming in the adjoint representation, the covariant derivative takes the analogous form

$$iD_\mu = i\partial_\mu + g_s A_\mu^a T^a, \quad (\text{A.9})$$

where the generators are given by

$$(T^a)_{bc} = -if^{abc}, \quad (\text{A.10})$$

with f^{abc} the totally antisymmetric structure constants of $SU(3)_c$. The generators satisfy the Lie algebra

$$[t^a, t^b] = if^{abc}t^c, \quad [T^a, T^b] = if^{abc}T^c. \quad (\text{A.11})$$

The gluon field-strength tensor is defined by

$$G_{\mu\nu} = G_{\mu\nu}^a t^a, \quad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c. \quad (\text{A.12})$$

Equivalently, in the fundamental representation it can be expressed through the commutator of two covariant derivatives (see e.g. Refs. [62, 120]),

$$G_{\mu\nu} = i[iD_\mu, iD_\nu]. \quad (\text{A.13})$$

Successive covariant derivatives acting on the field-strength tensor are therefore given by

$$D_\alpha G_{\mu\nu} = i[iD_\alpha, [iD_\mu, iD_\nu]], \quad (\text{A.14})$$

$$D_\beta D_\alpha G_{\mu\nu} = i[iD_\beta, [iD_\alpha, [iD_\mu, iD_\nu]]]. \quad (\text{A.15})$$

The following are the properties of the charge conjugation operator which are used in this work:

$$\begin{aligned} \bar{\psi}_R &= \bar{\psi} P_L, & \psi_R &= P_R \psi, \\ \bar{\psi}^c &= -\psi^T C^{-1}, & \psi^c &= C(\bar{\psi})^T, \\ \bar{\psi}_R^c &= -\psi^T (P_R)^T C^{-1}, & \psi_R^c &= C(P_L)^T (\bar{\psi})^T, \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} C\gamma^\mu C^{-1} &= -(\gamma^\mu)^T, & C\gamma_5 C^{-1} &= (\gamma_5)^T, \\ C\gamma^\mu \gamma^\nu C^{-1} &= (\gamma^\mu)^T (\gamma^\nu)^T, & CP_R \gamma^\mu C^{-1} &= -(\gamma^\mu)^T (P_L)^T, \\ C(P_L)^T (\gamma^\mu)^T (P_R)^T C^{-1} &= -P_L \gamma^\mu, & & \\ P_L &\equiv \left(\frac{1 - \gamma_5}{2} \right), & P_R &\equiv \left(\frac{1 + \gamma_5}{2} \right). \end{aligned} \quad (\text{A.17})$$

Appendix B

Supplementary Material for the Dimension-Seven Calculations

B.1 HQE Operator Basis

Within the HQE, the transition operator in [Eq. \(2.1\)](#) is expanded in terms of local operators of increasing dimension. This appendix summarizes the complete basis of local two-quark HQE operators up to dimension seven employed throughout this thesis. The operators are grouped according to their mass dimension and follow the conventions of [Appendix A](#).

Following the conventions of Refs. [\[121, 122\]](#), the hadronic matrix elements are parameterised as follows.

The dimension-five and dimension-six HQE parameters are defined by

$$\begin{aligned}
2m_B \mu_\pi^2(B) &= -\langle B(p_B) | \bar{b}_v(iD_\mu)(iD^\mu)b_v | B(p_B) \rangle, \\
2m_B \mu_G^2(B) &= \langle B(p_B) | \bar{b}_v(iD_\mu)(iD_\nu)(-i\sigma^{\mu\nu})b_v | B(p_B) \rangle, \\
2m_B \rho_D^3(B) &= \langle B(p_B) | \bar{b}_v(iD_\mu)(iv \cdot D)(iD^\mu)b_v | B(p_B) \rangle, \\
2m_B \rho_{LS}^3(B) &= \langle B(p_B) | \bar{b}_v(iD_\mu)(iv \cdot D)(iD_\nu)(-i\sigma^{\mu\nu})b_v | B(p_B) \rangle,
\end{aligned} \tag{B.1}$$

Here, μ_π^2 and μ_G^2 parameterise the kinetic and chromomagnetic operators at dimension five, respectively, whereas ρ_D^3 and ρ_{LS}^3 correspond to the Darwin and spin-orbit operators at dimension six. Moreover, $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, $v^\mu = p_B^\mu/m_B$, and b_v denotes the rephased full-QCD b -quark field, defined through

$$b(x) = e^{-im_b v \cdot x} b_v(x), \tag{B.2}$$

where the exponential factor removes the large momentum $m_b v^\mu$ carried by the heavy quark. The independent dimension-seven matrix elements are defined as

$$\begin{aligned}
2m_B r_G^4 &= \langle B(p_B) | \bar{b}_v[(iD_\mu), (iD_\nu)][(iD^\mu), (iD^\nu)]b_v | B(p_B) \rangle, \\
2m_B r_E^4 &= \langle B(p_B) | \bar{b}_v[(iv \cdot D), (iD_\mu)][(iv \cdot D), (iD^\mu)]b_v | B(p_B) \rangle, \\
2m_B s_B^4 &= \langle B(p_B) | \bar{b}_v[(iD_\mu), (iD_\alpha)][(iD^\mu), (iD_\beta)](-i\sigma^{\alpha\beta})b_v | B(p_B) \rangle, \\
2m_B s_E^4 &= \langle B(p_B) | \bar{b}_v[(iv \cdot D), (iD_\alpha)][(iv \cdot D), (iD_\beta)](-i\sigma^{\alpha\beta})b_v | B(p_B) \rangle, \\
2m_B s_{qB}^4 &= \langle B(p_B) | \bar{b}_v[(iD_\mu), [(iD^\mu), [(iD_\alpha), (iD_\beta)]]](-i\sigma^{\alpha\beta})b_v | B(p_B) \rangle.
\end{aligned} \tag{B.3}$$

Here, r_G^4 and r_E^4 parameterise the spin-independent field-strength and chromoelectric structures, respectively, while s_B^4 and s_E^4 describe spin-dependent structures involving two field strengths. The parameter s_{qB}^4 instead corresponds to a spin-dependent structure involving covariant derivatives of the field-strength tensor.

Furthermore, the matrix elements r_G^4 and r_E^4 can be decomposed into colour-singlet and colour-octet components using the $SU(3)_c$ identity $\{t^a, t^b\} = \delta^{ab}/N_c + d^{abc}t^c$. The corresponding matrix elements are defined by

$$\begin{aligned} 2m_B r_G^{(1)} &= \langle B(p_B) | \bar{b}_v b_v G_{\mu\nu}^a G^{\mu\nu a} | B(p_B) \rangle, \\ 2m_B r_G^{(8)} &= \langle B(p_B) | \bar{b}_v t^c b_v d^{abc} G_{\mu\nu}^a G^{\mu\nu b} | B(p_B) \rangle, \\ 2m_B r_E^{(1)} &= \langle B(p_B) | \bar{b}_v b_v v_\alpha v_\beta G_{\alpha\nu}^a G^{\beta\nu a} | B(p_B) \rangle, \\ 2m_B r_E^{(8)} &= \langle B(p_B) | \bar{b}_v t^c b_v v_\alpha v_\beta d^{abc} G_{\alpha\nu}^a G^{\beta\nu b} | B(p_B) \rangle. \end{aligned} \tag{B.4}$$

The colour decomposition is particularly relevant for distinguishing semileptonic from non-leptonic decays at leading order QCD. In the semileptonic case, the colour structure involves a product of two colour generators, such that the colour-singlet and colour-octet matrix elements enter only through the combination implied by the identity above. In non-leptonic decays, additional colour contractions allow the colour-singlet matrix elements $r_G^{(1)}$ and $r_E^{(1)}$ to appear separately.

B.2 Loop Integrals

The two-loop integrals arising in the matching of the dimension-seven and dimension six two-quark operators can be reduced, by means of IBP identities, to a small set of master integrals. We define the generic scalar integral as

$$\mathcal{I}_{n_1 n_2 n_3} = \int \frac{d^4 l_1}{(2\pi)^4} \int \frac{d^4 l_2}{(2\pi)^4} \frac{1}{[l_1^2 - m_1^2]^{n_1} [l_2^2 - m_2^2]^{n_2} [(l_1 + l_2 - p_b)^2 - m_3^2]^{n_3}}, \tag{B.5}$$

where $n_i \in \mathbb{N}_0$ and the Feynman $+i\varepsilon$ prescription is understood for all propagators. The complete set of master integrals required for the present calculation is

$$\{\mathcal{I}_{111}, \mathcal{I}_{211}, \mathcal{I}_{121}, \mathcal{I}_{112}, \mathcal{I}_{011}, \mathcal{I}_{101}, \mathcal{I}_{110}\}. \tag{B.6}$$

The first four integrals correspond to the scalar sunrise topology with three propagators of masses m_1 , m_2 , and m_3 [123, 124]. Since increasing the power of a propagator is equivalent to differentiating with respect to the corresponding mass squared, they satisfy

$$\mathcal{I}_{211} = \frac{\partial}{\partial m_1^2} \mathcal{I}_{111}, \quad \mathcal{I}_{121} = \frac{\partial}{\partial m_2^2} \mathcal{I}_{111}, \quad \mathcal{I}_{112} = \frac{\partial}{\partial m_3^2} \mathcal{I}_{111}. \tag{B.7}$$

The remaining three master integrals factorize into products of scalar tadpole integrals. For example,

$$\mathcal{I}_{101} = \int \frac{d^4 l_1}{(2\pi)^4} \frac{1}{l_1^2 - m_1^2 + i\varepsilon} \int \frac{d^4 l_2}{(2\pi)^4} \frac{1}{l_2^2 - m_3^2 + i\varepsilon}, \tag{B.8}$$

which follows after the change of variables $l_1 + l_2 - p_b \rightarrow l_2$. Analogous relations hold for $\mathcal{I}_{011}$ and $\mathcal{I}_{110}$. Since the imaginary part of a product of tadpole integrals vanishes,

$$\text{Im } \mathcal{I}_{101} = \text{Im } \mathcal{I}_{110} = \text{Im } \mathcal{I}_{011} = 0, \tag{B.9}$$

these integrals do not contribute to the absorptive part of the transition operator.

Consequently, it is sufficient to determine the discontinuity of the single master integral $\mathcal{I}_{111}$ in the physical region $p_b^2 \geq (m_1 + m_2 + m_3)^2$, from which the remaining master integrals follow through differentiation. Using the result of Ref. [74], evaluated in four dimensions, one finds

$$\text{Im } \mathcal{I}_{111} = \frac{1}{256\pi^3} \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} dt \frac{\sqrt{\lambda(t, m_2^2, m_3^2) \lambda(s, t, m_1^2)}}{ts}, \quad (\text{B.10})$$

where $s = p_b^2$ and

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \quad (\text{B.11})$$

denotes the Källén function. The master integral $\mathcal{I}_{111}$ admits a closed analytical representation only when at most two internal masses are non-vanishing. This is sufficient for all decay channels considered in this work except for the $b \rightarrow c\bar{c}s$ transition, where all three internal propagators carry finite masses. In this case, the analytical expression is obtained only after combining the two-quark contribution with the corresponding four-quark mixing terms, leading to the cancellation of the IR logarithms. Using the result of Ref. [100], one finds

$$\begin{aligned} \left[\mathcal{M}_{112}(\rho, \eta) - \sqrt{1-4\rho} \log(\eta) \right]_{\eta \rightarrow 0} &= 2(1-\rho) \log \left(\frac{1 + \sqrt{1-4\rho}}{1 - \sqrt{1-4\rho}} \right) \\ &\quad + \sqrt{1-4\rho} \left[1 + 2\log(\rho) - 4\log(\sqrt{1-4\rho}) \right], \end{aligned} \quad (\text{B.12})$$

where

$$\mathcal{M}_{112}(\rho, \eta) = 256\pi^3 \text{Im } \mathcal{I}_{112} \Big|_{m_1^2=m_2^2=m_c^2, m_3^2=m_q^2, p_b^2=m_b^2}, \quad (\text{B.13})$$

with $\rho = m_c^2/m_b^2$ and $\eta = m_q^2/m_b^2$.

Finally, the gluon corrections to the four-quark operators reduce to one-loop tadpole integrals. Throughout the calculations we employ the standard dimensional-regularisation result

$$\mu^{2\epsilon} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2 + i0)^b} = i(-1)^{-b} \frac{\mu^{2\epsilon}}{(4\pi)^{D/2}} (m^2)^{D/2-b} \frac{\Gamma(b - \frac{D}{2})}{\Gamma(b)}, \quad (\text{B.14})$$

where $D = 4 - 2\epsilon$ and μ denotes the renormalisation scale introduced to keep the strong coupling dimensionless in dimensional regularisation.

B.3 Trace Formulas for Colour-Decomposed Matrix Elements

The trace formulas provide the systematic translation between general hadronic matrix elements and the independent HQE parameters defined in Eqs. (B.1), (B.3) and (B.4) [87, 125, 126]. While the complete set of trace formulas at $\mathcal{O}(1/m_b^4)$

for the combined matrix elements r_G^4 and r_E^4 was derived in [79, 125], the colour decomposition of these matrix elements requires corresponding formulas for the separate colour-singlet and colour-octet components $r_G^{(1)}$, $r_G^{(8)}$, $r_E^{(1)}$, and $r_E^{(8)}$ presented in Eq. (B.4). In this appendix, we derive these relations.

We start with the colour singlet case and write the most general matrix element

$$\langle B(v) | \bar{b}_{v,j} b_{v,i} (G_{\mu\nu}^a G_{\alpha\beta}^a) | B(v) \rangle = \mathcal{M}_{ij,\mu\nu\alpha\beta}, \quad (\text{B.15})$$

where i, j are Dirac indices and μ, ν, α, β are Lorentz indices. Note that no colour indices appear, since everything is colour singlet. Concerning the Lorentz indices, we have to construct this from the metric tensor g and the velocity vector v , and possibly also from Lorentz indices related to the Dirac matrices.

For the trivial case of Dirac matrices, we have the four indices generated by the two structures

$$R_{\mu\nu\alpha\beta}^{(G)} = g_{\mu\alpha} g_{\nu\beta}^\perp - g_{\nu\alpha} g_{\mu\beta}^\perp, \quad (\text{B.16})$$

$$R_{\mu\nu\alpha\beta}^{(E)} = g_{\mu\alpha} v_\nu v_\beta - g_{\nu\alpha} v_\mu v_\beta - g_{\mu\beta}^\perp v_\alpha v_\nu + g_{\nu\beta}^\perp v_\alpha v_\mu, \quad (\text{B.17})$$

with $g_{\mu\nu}^\perp = g_{\mu\nu} - v_\mu v_\nu$, which are antisymmetric under $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$ and symmetric under $(\mu\nu) \leftrightarrow (\alpha\beta)$. Dirac matrices that carry Lorentz indices are γ_μ and $\sigma_{\mu\nu}$. Since we are at highest order, γ_μ is equivalent to v_μ , while $\sigma_{\mu\nu}$ is antisymmetric in μ and ν and no scalar operator can be formed with a single $\sigma_{\mu\nu}$. In this way we obtain

$$\mathcal{M}_{ij,\mu\nu\alpha\beta} = \left(\frac{\not{v} + 1}{2} \right)_{ij} \left[a_G^{(1)} R_{\mu\nu\alpha\beta}^{(G)} + a_E^{(1)} R_{\mu\nu\alpha\beta}^{(E)} \right], \quad (\text{B.18})$$

where $a_G^{(1)}$ and $a_E^{(1)}$ are determined from Eqs. (B.16) and (B.17) in terms of $r_G^{(1)}$ and $r_E^{(1)}$, and we get

$$2M_B r_G^{(1)} = 12a_G^{(1)}, \quad (\text{B.19})$$

$$2M_B r_E^{(1)} = 2 \left[6a_G^{(1)} + 3a_E^{(1)} \right]. \quad (\text{B.20})$$

The formulae for the octet part look completely the same, with the replacement superscript $(1) \rightarrow (8)$.

Appendix C

Supplementary Analytical Results for the BSM Darwin Term

This appendix collects the supplementary analytical results for the generalised effective Hamiltonian discussed in [Section 3.3](#). Owing to the length of the complete expressions, machine-readable versions of all matching coefficients are provided in the ancillary files accompanying Ref. [89], to which the interested reader is referred. The notation and conventions adopted throughout this appendix are the same as those introduced in [Chapter 3](#).

C.1 Results for the $b \rightarrow u\bar{c}s$ Transition

For completeness, we collect here the matching coefficients for the CKM-suppressed transition $b \rightarrow u\bar{c}s$. As discussed in [Section 3.2](#), the cancellation of the IR divergences in this channel requires the inclusion of the WA contribution in addition to the PI and WE topologies (see [Figure 3.1](#)). The corresponding WA matching coefficients are given in [Section C.3](#).

Dimension-three contribution (Γ_3): The independent non-vanishing coefficient functions $P_{i,j}^{(ucs)}(\rho)$ are

$$P_{1,1}^{(ucs)}(\rho) = 1 - 8\rho - 12\rho^2 \log \rho + 8\rho^3 - \rho^4, \quad (\text{C.1})$$

$$P_{1,7}^{(ucs)}(\rho) = \sqrt{\rho}(-1 - 9\rho + 9\rho^2 + \rho^3 - 6\rho(1 + \rho) \log \rho). \quad (\text{C.2})$$

The remaining non-vanishing coefficient functions are obtained from

$$\begin{aligned} P_{2,2}^{(ucs)} &= P_{3,3}^{(ucs)} = P_{4,4}^{(ucs)} = P_{1,1}^{(ucs)}, \\ 4P_{5,5}^{(ucs)} &= 4P_{6,6}^{(ucs)} = 4P_{7,7}^{(ucs)} = 4P_{8,8}^{(ucs)} = P_{1,1}^{(ucs)}, \\ \frac{1}{12}P_{9,9}^{(ucs)} &= \frac{1}{12}P_{10,10}^{(ucs)} = P_{1,1}^{(ucs)}, \\ P_{3,4}^{(ucs)} &= 4P_{5,6}^{(ucs)} = 4P_{7,8}^{(ucs)} = \frac{1}{12}P_{9,10}^{(ucs)} = P_{1,1}^{(ucs)}, \\ P_{1,8}^{(ucs)} &= P_{2,8}^{(ucs)} = P_{3,5}^{(ucs)} = P_{4,5}^{(ucs)} = P_{1,7}^{(ucs)}, \\ -\frac{1}{12}P_{1,9}^{(ucs)} &= -\frac{1}{12}P_{1,10}^{(ucs)} = -\frac{1}{12}P_{2,10}^{(ucs)} = P_{1,7}^{(ucs)}, \\ P_{i,j}^{(ucs)} &= P_{i-1,j+1}^{(ucs)}, \quad \text{for even } i \text{ and odd } j, \end{aligned} \quad (\text{C.3})$$

while all other coefficient functions vanish identically.

Dimension-five contribution (Γ_5): The independent non-vanishing chromomagnetic coefficient functions $G_{i,j}^{(ucs)}(\rho)$ are

$$G_{1,1}^{(ucs)}(\rho) = -\frac{1}{2}(5\rho^4 - 24\rho^3 + 24\rho^2 + 12\rho^2 \log \rho - 8\rho + 3), \quad (C.4)$$

$$G_{1,2}^{(ucs)}(\rho) = -\frac{1}{2}(19 + 16\rho - 24\rho^2 - 16\rho^3 + 5\rho^4 + 12\rho(4 + \rho) \log \rho), \quad (C.5)$$

$$G_{1,7}^{(ucs)}(\rho) = \frac{1}{2}\sqrt{\rho}(-5 + 3\rho - 3\rho^2 + 5\rho^3 - 6\rho(1 + \rho) \log \rho), \quad (C.6)$$

$$G_{1,8}^{(ucs)}(\rho) = \frac{3}{2}\sqrt{\rho}(-9 + 7\rho + \rho^2 + \rho^3 - 2(2 + 3\rho + \rho^2) \log \rho), \quad (C.7)$$

$$G_{1,9}^{(ucs)}(\rho) = 2\sqrt{\rho}(-17 + 15\rho + 9\rho^2 - 7\rho^3 + 6\rho(-5 + 3\rho) \log \rho), \quad (C.8)$$

$$G_{2,8}^{(ucs)}(\rho) = \frac{1}{2}\sqrt{\rho}(11 - 9\rho - 3\rho^2 + \rho^3 - 6(-3 + \rho)\rho \log \rho), \quad (C.9)$$

$$G_{2,10}^{(ucs)}(\rho) = -2\sqrt{\rho}(1 - 3\rho - 9\rho^2 + 11\rho^3 + 6(1 - 3\rho)\rho \log \rho), \quad (C.10)$$

$$G_{3,4}^{(ucs)}(\rho) = \frac{1}{2}(13 - 40\rho + 24\rho^2 + 8\rho^3 - 5\rho^4 - 12\rho^2 \log \rho), \quad (C.11)$$

$$G_{3,6}^{(ucs)}(\rho) = \frac{1}{6}\sqrt{\rho}(-17 - 9\rho + 9\rho^2 + 17\rho^3 - 6(-2 + 9\rho + 3\rho^2) \log \rho), \quad (C.12)$$

$$G_{7,10}^{(ucs)}(\rho) = -4(-4 + 3\rho + \rho^3 - 6\rho \log \rho), \quad (C.13)$$

$$G_{8,10}^{(ucs)}(\rho) = 8(1 - \rho)^3, \quad (C.14)$$

$$G_{9,9}^{(ucs)}(\rho) = -2(25 - 72\rho + 120\rho^2 - 88\rho^3 + 15\rho^4 + 36\rho^2 \log \rho), \quad (C.15)$$

$$G_{9,10}^{(ucs)}(\rho) = -2(25 + 72\rho - 72\rho^2 - 40\rho^3 + 15\rho^4 + 12\rho(8 + 3\rho) \log \rho), \quad (C.16)$$

$$G_{10,10}^{(ucs)}(\rho) = 2(23 - 72\rho + 24\rho^2 + 40\rho^3 - 15\rho^4 - 36\rho^2 \log \rho). \quad (C.17)$$

The remaining non-vanishing coefficient functions are obtained from the relations

$$\begin{aligned} G_{2,2}^{(ucs)} &= G_{3,3}^{(ucs)} = 4G_{6,6}^{(ucs)} = 4G_{8,8}^{(ucs)} = G_{1,1}^{(ucs)}, & G_{1,10}^{(ucs)} &= -12G_{1,8}^{(ucs)}, \\ G_{4,4}^{(ucs)} &= 4G_{5,5}^{(ucs)} = 4G_{5,6}^{(ucs)} = 4G_{7,7}^{(ucs)} = 4G_{7,8}^{(ucs)} = G_{3,4}^{(ucs)}, & G_{3,5}^{(ucs)} &= G_{4,6}^{(ucs)} = G_{1,7}^{(ucs)}, \\ G_{i,j}^{(ucs)} &= G_{i-1,j+1}^{(ucs)} \quad \text{for even } i \text{ and odd } j, \end{aligned} \quad (C.18)$$

while all other coefficient functions vanish identically.

Dimension-six contribution (Γ_6): The independent non-vanishing Darwin coefficient functions $D_{i,j}^{(ucs)}(\rho)$ are

$$D_{1,1}^{(ucs)}(\rho) = \frac{2}{3}(1 - \rho)(9 + 11\rho - 25\rho^2 + 5\rho^3 - 24(1 - \rho^2) \log(1 - \rho) - 12\rho^2 \log \rho), \quad (C.19)$$

$$D_{1,2}^{(ucs)}(\rho) = -\frac{2}{3}(41 - 26\rho + 18\rho^2 - 38\rho^3 + 5\rho^4 + 48(1 - \rho)^2(1 + \rho) \log(1 - \rho) + 12(2 + 5\rho + 2\rho^2 - 2\rho^3) \log \rho), \quad (C.20)$$

$$D_{1,7}^{(ucs)}(\rho) = \frac{2}{3}\sqrt{\rho}\left(-8 + 27\rho - 24\rho^2 + 5\rho^3 + 12(1 - \rho)^2 \log(1 - \rho) - 6\rho(-2 + \rho) \log \rho\right), \quad (C.21)$$

$$D_{1,8}^{(ucs)}(\rho) = \frac{2}{3}\sqrt{\rho}\left(7 + 6\rho - 15\rho^2 + 2\rho^3 + 24(1 - \rho)^2 \log(1 - \rho) + 6(1 + 4\rho - 2\rho^2) \log \rho\right), \quad (C.22)$$

$$D_{1,9}^{(ucs)}(\rho) = -8\sqrt{\rho}(\rho(7 - 8\rho + \rho^2 - 6(-2 + \rho) \log \rho) + 12(1 - \rho)^2 \log(1 - \rho)), \quad (C.23)$$

$$D_{2,8}^{(ucs)}(\rho) = -\frac{2}{3}\sqrt{\rho}(-4 + 3\rho + \rho^3 - 12(1 - \rho)^2 \log(1 - \rho) + 6\rho(-2 + \rho) \log \rho), \quad (C.24)$$

$$D_{2,10}^{(ucs)}(\rho) = -8\sqrt{\rho}\left(-4 + 17\rho - 16\rho^2 + 3\rho^3 + 12(1 - \rho)^2 \log(1 - \rho) - 6\rho(-2 + \rho) \log \rho\right), \quad (C.25)$$

$$D_{3,4}^{(ucs)}(\rho) = -\frac{2}{3}(-51 + 94\rho - 42\rho^2 - 6\rho^3 + 5\rho^4 + 24(1 - \rho)^2(3 + \rho) \log(1 - \rho) - 12(-1 + \rho^2 + \rho^3) \log \rho), \quad (C.26)$$

$$D_{3,6}^{(ucs)}(\rho) = 4\sqrt{\rho}(1 + 3\rho - 5\rho^2 + \rho^3 + 4(1 - \rho)^2 \log(1 - \rho) + 2(1 + 2\rho - \rho^2) \log \rho), \quad (C.27)$$

$$D_{4,4}^{(ucs)}(\rho) = -\frac{2}{3}\left(-59 + 112\rho - 66\rho^2 + 8\rho^3 + 5\rho^4 + 48(1 - \rho)^2 \log(1 - \rho) - 24\rho^2 \log \rho\right), \quad (C.28)$$

$$D_{7,8}^{(ucs)}(\rho) = -\frac{1}{6}(-55 + 106\rho - 54\rho^2 - 2\rho^3 + 5\rho^4 + 24(1 - \rho)^2(3 + \rho) \log(1 - \rho) - 12(-1 + \rho^2 + \rho^3) \log \rho), \quad (C.29)$$

$$D_{7,10}^{(ucs)}(\rho) = \frac{8}{3}\left(22 - 27\rho + 12\rho^2 - 7\rho^3 - 6(1 - \rho)^3 \log(1 - \rho) + 3(1 + 5\rho + 3\rho^2 - \rho^3) \log \rho\right), \quad (C.30)$$

$$D_{8,8}^{(ucs)}(\rho) = \frac{1}{6}(1 - \rho)(13 + 3\rho - 21\rho^2 + 5\rho^3 - 24(1 - \rho^2) \log(1 - \rho) - 12\rho^2 \log \rho), \quad (C.31)$$

$$D_{8,10}^{(ucs)}(\rho) = \frac{4}{3}(23 - 51\rho + 45\rho^2 - 17\rho^3 - 12(1 - \rho)^3 \log(1 - \rho) + 6(3 - \rho)\rho^2 \log \rho), \quad (C.32)$$

$$D_{9,9}^{(ucs)}(\rho) = -\frac{8}{3}(7 - 72\rho + 162\rho^2 - 112\rho^3 + 15\rho^4 + 48(1 - \rho)^2(1 + 2\rho)\log(1 - \rho) + 24(3 - 2\rho)\rho^2\log\rho), \quad (\text{C.33})$$

$$D_{9,10}^{(ucs)}(\rho) = -\frac{8}{3}(11 + 102\rho - 66\rho^2 - 62\rho^3 + 15\rho^4 + 24(1 - \rho)^2(7 + 5\rho)\log(1 - \rho) + 12(5 + 10\rho + 3\rho^2 - 5\rho^3)\log\rho), \quad (\text{C.34})$$

$$D_{10,10}^{(ucs)}(\rho) = -\frac{8}{3}(-131 + 234\rho - 108\rho^2 - 10\rho^3 + 15\rho^4 + 24(1 - \rho)^2(5 + \rho)\log(1 - \rho) - 12\rho^2(3 + \rho)\log\rho). \quad (\text{C.35})$$

The remaining non-vanishing coefficient functions are obtained from

$$\begin{aligned} D_{2,2}^{(ucs)} &= D_{3,3}^{(ucs)} = 4D_{6,6}^{(ucs)} = D_{1,1}^{(ucs)}, & D_{1,10}^{(ucs)} &= -12D_{1,8}^{(ucs)}, \\ D_{3,5}^{(ucs)} &= D_{4,6}^{(ucs)} = D_{1,7}^{(ucs)}, & D_{5,5}^{(ucs)} &= D_{7,7}^{(ucs)} = \frac{1}{4}D_{4,4}^{(ucs)}, & D_{5,6}^{(ucs)} &= \frac{1}{4}D_{3,4}^{(ucs)}, \\ D_{i,j}^{(ucs)} &= D_{i-1,j+1}^{(ucs)}, & \text{for even } i \text{ and odd } j, \end{aligned} \quad (\text{C.36})$$

while all other coefficient functions vanish identically.

C.2 Results for the $b \rightarrow u\bar{u}d$ Transition

The dimension-three and dimension-five matching coefficients for the $b \rightarrow u\bar{u}d$ transition are obtained directly from the corresponding $b \rightarrow c\bar{u}d$ results by taking the massless limit $\rho \rightarrow 0$. The Darwin contribution is an exception, as the IR divergences arising from the expansion of the q_1 -quark propagator require a dedicated matching calculation in the massless case. The corresponding coefficient functions are presented below.

Dimension-six contribution (Γ_6): The independent non-vanishing Darwin coefficient functions $D_{i,j}^{(uud)}$ are

$$D_{1,1}^{(uud)}(\rho) = 6, \quad D_{1,2}^{(uud)}(\rho) = -\frac{34}{3}, \quad D_{3,4}^{(uud)}(\rho) = \frac{110}{3}, \quad D_{4,4}^{(uud)}(\rho) = \frac{118}{3}, \quad (\text{C.37})$$

$$D_{5,5}^{(uud)}(\rho) = \frac{59}{6}, \quad D_{5,6}^{(uud)}(\rho) = \frac{55}{6}, \quad D_{6,6}^{(uud)}(\rho) = \frac{3}{2}, \quad D_{7,8}^{(uud)}(\rho) = \frac{21}{2}, \quad (\text{C.38})$$

$$D_{7,10}^{(uud)}(\rho) = 48, \quad D_{8,8}^{(uud)}(\rho) = \frac{13}{6}, \quad D_{8,10}^{(uud)}(\rho) = \frac{92}{3}, \quad D_{9,9}^{(uud)}(\rho) = -\frac{56}{3}, \quad (\text{C.39})$$

$$D_{9,10}^{(uud)}(\rho) = 120, \quad D_{10,10}^{(uud)}(\rho) = \frac{1048}{3}. \quad (\text{C.40})$$

The remaining non-vanishing coefficients follow from the relations

$$\begin{aligned}
D_{3,3}^{(uud)} &= D_{1,1}^{(uud)}, & D_{7,7}^{(uud)} &= D_{5,5}^{(uud)}, & D_{2,2}^{(uud)} &= D_{1,1}^{(uud)}, \\
D_{i,j}^{(uud)} &= D_{i-1,j+1}^{(uud)} & \text{for even } i \text{ and odd } j,
\end{aligned} \tag{C.41}$$

while all other coefficient functions vanish identically.

C.3 Results for the WA Topology

In this appendix we collect the matching coefficients for the WA topology induced by the generalised effective Hamiltonian in Eq. (3.1). As discussed in Section 3.2, these contributions are required to complete the dimension-six four-quark matching and to ensure the cancellation of the IR divergences in the $b \rightarrow u\bar{c}s$ and $b \rightarrow u\bar{u}d$ channels.

Following the notation of Ref. [81], the WA contribution to the transition operator for internal quark flavours $q_2 = c, u$ and $q_3 = d, s$ is parametrised as

$$\text{Im } \mathcal{T}_{\text{WA}}^{q_2 q_3} = \frac{G_F^2 m_b^2}{6\pi} |V_{ub}|^2 |V_{q_2 q_3}|^2 (1 - \rho)^2 \left[\sum_{m,n=1}^{20} \mathcal{C}_m \mathcal{C}_n^* \mathcal{N}'_{mn} A_{mn}^{\text{WA}, q_2 q_3} \right], \tag{C.42}$$

where the colour factor $\mathcal{N}'_{mn}$ is given by $\mathcal{N}'_{mn} = N_c$ if both indices m and n are odd, and $\mathcal{N}'_{mn} = 1$ otherwise. For convenience, the primed Wilson coefficients are incorporated into the same index according to

$$\mathcal{C}_m \equiv \mathcal{C}'_{m-10}, \quad m > 10. \tag{C.43}$$

The independent non-vanishing coefficient functions $A_{mn}^{\text{WA}, q_2 q_3}$ are given below for the $q_2 q_3 = cs$ channel, retaining the full charm-mass dependence. The corresponding expressions for $q_2 q_3 = ud$ follow from the massless limit, $\rho \rightarrow 0$.

$$A_{1,1}^{\text{WA},cs} = -\left[\mathcal{O}_1\left(1 + \frac{\rho}{2}\right) - \mathcal{O}_2(1 + 2\rho)\right], \quad A_{1,3}^{\text{WA},cs} = -\left[\mathcal{O}'_5\left(1 + \frac{\rho}{2}\right) - \mathcal{O}'_6(1 + 2\rho)\right], \tag{C.44}$$

$$A_{1,5}^{\text{WA},cs} = -\frac{3}{2}\mathcal{O}'_6\sqrt{\rho}, \quad A_{1,7}^{\text{WA},cs} = -\frac{3}{2}\mathcal{O}_2\sqrt{\rho}, \quad A_{1,9}^{\text{WA},cs} = -6\sqrt{\rho}[\mathcal{O}_1 - \mathcal{O}_2], \tag{C.45}$$

$$A_{3,5}^{\text{WA},cs} = -\frac{3}{2}\mathcal{O}'_2\sqrt{\rho}, \quad A_{3,7}^{\text{WA},cs} = -\frac{3}{2}\mathcal{O}_6\sqrt{\rho}, \quad A_{3,9}^{\text{WA},cs} = -6\sqrt{\rho}[\mathcal{O}_5 - \mathcal{O}_6], \tag{C.46}$$

$$A_{5,5}^{\text{WA},cs} = \frac{3}{2}\mathcal{O}'_2, \quad A_{5,7}^{\text{WA},cs} = \frac{3}{2}\mathcal{O}_6, \quad A_{9,9}^{\text{WA},cs} = -8(\mathcal{O}_1 - \mathcal{O}_2)(1 + 2\rho), \tag{C.47}$$

where the $\Delta B = 0$ operators $\mathcal{O}_i^{(\prime)}$ are defined in Eqs. (3.12) to (3.15). The remaining non-vanishing coefficient functions follow from the symmetry relations

$$A_{3,3}^{\text{WA},cs} = A_{1,1}^{\text{WA},cs} \Big|_{\mathcal{O}_i \leftrightarrow \mathcal{O}'_i}, \quad A_{7,7}^{\text{WA},cs} = A_{5,5}^{\text{WA},cs} \Big|_{\mathcal{O}_i \leftrightarrow \mathcal{O}'_i}, \tag{C.48}$$

$$A_{m,n}^{\text{WA},cs} = \begin{cases} A_{m,n-1}^{\text{WA},cs}, & m \text{ odd}, n \text{ even}, \\ A_{m-1,n+1}^{\text{WA},cs}, & m \text{ even}, n \text{ odd}, \end{cases} \tag{C.49}$$

$$A_{m,n}^{\text{WA},cs} = A_{m-1,n-1}^{\text{WA},cs} \Big|_{\mathcal{O}_i^{(\prime)} \rightarrow \left(\frac{\mathcal{O}_i^{(\prime)}}{N_c} + 2\mathcal{O}_{i+2}^{(\prime)} \right)}, \quad m, n \text{ even}, \quad (\text{C.50})$$

$$A_{m,n}^{\text{WA},cs} = A_{n,m}^{\text{WA},cs} \Big|_{\mathcal{O}_i \leftrightarrow \mathcal{O}_i'}, \quad m \neq n, \quad (\text{C.51})$$

$$A_{m,n}^{\text{WA},cs} = A_{m+10,n+10}^{\text{WA},cs} \Big|_{\mathcal{O}_i \leftrightarrow \mathcal{O}_i'}, \quad m, n \in \{1, \dots, 10\}, \quad (\text{C.52})$$

while all other coefficient functions vanish identically.

C.4 Contribution of SM QCD-Penguin Operators up to Dimension-Six

As discussed in [Section 3.1](#), the SM QCD-penguin operators are related to the generalised operator basis through the Fierz identities in [Eqs. \(3.8\) to \(3.10\)](#). Consequently, once the matching coefficients $\mathcal{F}_{i,j}^{(q_1 q_2 q_3)}$ have been determined (see [Eqs. \(3.17\) and \(3.19\)](#)), the corresponding SM penguin contributions follow directly without requiring an independent calculation.

The interference between the current-current operators $Q_{1,2}^{qqq3}$ and the QCD-penguin operators $Q_{3,\dots,6}^{qqq3}$ is given by

$$\begin{aligned} \frac{\Gamma_{\text{PO}}^{\text{int}}(b \rightarrow q\bar{q}q_3)}{\Gamma_0} &= 2 \text{Re}(\xi_{qq3} \xi_{tq3}^*) \left[\right. \\ &\quad C_1 \left(C_3 \mathcal{F}_{1,2}^{(qqq3)} + C_4 \mathcal{F}_{1,1}^{(qqq3)} - 2C_5 \mathcal{F}_{1,6}^{(qqq3)} - 2C_6 \mathcal{F}_{1,5}^{(qqq3)} \right) \\ &\quad \left. + C_2 \left(C_3 \mathcal{F}_{2,2}^{(qqq3)} + C_4 \mathcal{F}_{2,1}^{(qqq3)} - 2C_5 \mathcal{F}_{2,6}^{(qqq3)} - 2C_6 \mathcal{F}_{2,5}^{(qqq3)} \right) \right]. \end{aligned} \quad (\text{C.53})$$

where $q = u, c$, $q_3 = d, s$, and we have introduced the shorthand $\xi_{xy} \equiv V_{xb} V_{xy}^*$.

The contribution arising from the double insertion of the QCD-penguin operators takes the form

$$\begin{aligned} \frac{\Gamma_{\text{PO}}^{\text{quad}}(b \rightarrow q\bar{q}q_3)}{\Gamma_0} &= |\xi_{tq3}|^2 \left[\left(C_3^2 \mathcal{F}_{2,2}^{(qqq3)} + 2C_3 C_4 \mathcal{F}_{2,1}^{(qqq3)} + C_4^2 \mathcal{F}_{1,1}^{(qqq3)} \right) \right. \\ &\quad + 4 \left(C_5^2 \mathcal{F}_{6,6}^{(qqq3)} + 2C_5 C_6 \mathcal{F}_{6,5}^{(qqq3)} + C_6^2 \mathcal{F}_{5,5}^{(qqq3)} \right) \\ &\quad \left. - 4 \left(C_3 C_5 \mathcal{F}_{2,6}^{(qqq3)} + C_3 C_6 \mathcal{F}_{2,5}^{(qqq3)} + C_4 C_5 \mathcal{F}_{1,6}^{(qqq3)} + C_4 C_6 \mathcal{F}_{1,5}^{(qqq3)} \right) \right]. \end{aligned} \quad (\text{C.54})$$

The quadratic contribution is valid for $q = u, d, s, c$. Since the masses of the light quarks are neglected, the corresponding coefficient functions satisfy

$$\mathcal{F}_{i,j}^{ddq3} = \mathcal{F}_{i,j}^{ssq3} = \mathcal{F}_{i,j}^{uuq3}. \quad (\text{C.55})$$

As a consistency check, we have verified that the leading-power contributions obtained by inserting the functions $P_{i,j}^{(ccs)}(\rho)$ into Eqs. (C.53) and (C.54) reproduce the known SM results of Ref. [91]. Throughout this section the Wilson coefficients are understood to be the real-valued SM coefficients.

Appendix D

Supplementary Material for the B -Mesogenesis Analysis

The free quark propagator in coordinate space is given by

$$\begin{aligned} S_{jn}^{(q)}(x, y) &= \delta_{jn} S_0^{(q)}(x - y), \\ S_0^{(q)}(x - y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}, \\ \left(S_0^{(q)}\right)^T(x - y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{p_\mu (\gamma^\mu)^T + m}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}. \end{aligned} \quad (\text{D.1})$$

Furthermore, the following one-loop tensor integral frequently appears in the evaluation of the dimension-six four-quark contributions. Its absorptive part is given by (see, e.g., Appendices B and C of Ref. [62])

$$\begin{aligned} &\text{Im} \left(i \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu - k_\mu p_\nu}{[k^2 - m_1^2][(p - k)^2 - m_2^2]} \right) = \\ &\frac{p^2 \sqrt{\lambda(1, r_1, r_2)}}{192\pi} \left(g_{\mu\nu} \lambda(1, r_1, r_2) - 2 \frac{p_\mu p_\nu}{p^2} (2(r_1 - r_2)^2 - (1 + r_1 + r_2)) \right), \end{aligned} \quad (\text{D.2})$$

where $r_i = m_i^2/p^2$ and $\lambda(a, b, c) = (a - b - c)^2 - 4bc$ denotes the Källén function.

The matrix elements of the four-quark operators arising in the B -Mesogenesis calculations are expressed in terms of the operator basis¹

$$\begin{aligned} \tilde{\mathcal{O}}_1^q &= 4 (\bar{q}^i \gamma^\mu P_R b^i) (\bar{b}^k \gamma_\mu P_R q^k), \\ \tilde{\mathcal{O}}_2^q &= 4 (\bar{q}^i P_L b^i) (\bar{b}^k P_R q^k), \\ \tilde{\mathcal{O}}_3^q &= 4 (\bar{q}^i \gamma^\mu P_R t_{ik}^a b^k) (\bar{b}^r \gamma_\mu P_R t_{rm}^a q^m), \\ \tilde{\mathcal{O}}_4^q &= 4 (\bar{q}^i P_L t_{ik}^a b^k) (\bar{b}^r P_R t_{rm}^a q^m), \end{aligned} \quad (\text{D.3})$$

where $q = u, d$. Their hadronic matrix elements are parameterized as [38, 127]

$$\begin{aligned} \langle B | \tilde{\mathcal{O}}_{1,2}^q | B \rangle &= f_B^2 M_B^2 B_{1,2}^q, \\ \langle B | \tilde{\mathcal{O}}_{3,4}^q | B \rangle &= f_B^2 M_B^2 B_{3,4}^q, \end{aligned} \quad (\text{D.4})$$

where $B_{1,2}^q$ and $B_{3,4}^q$ denote the corresponding bag parameters. Throughout this work, isospin symmetry is assumed,

$$B_i^u = B_i^d, \quad i = 1, \dots, 4. \quad (\text{D.5})$$

¹The operator basis differs from the corresponding SM basis only by the replacement $P_L \leftrightarrow P_R$ (see Eq. (3.15)). Owing to parity invariance of QCD, the parametrisation of the hadronic matrix elements is identical to that of the SM operators.

D.1 Input Parameters

For the bottom and charm quark masses we use the values obtained from the global fit of Ref. [128],

$$m_b^{\text{kin}}(1 \text{ GeV}) = (4.573 \pm 0.012) \text{ GeV}, \quad \bar{m}_c(2 \text{ GeV}) = (1.090 \pm 0.010) \text{ GeV}. \quad (\text{D.6})$$

The B -meson decay constants are taken from the lattice-QCD averages of Ref. [129],

$$f_B = (0.1900 \pm 0.0013) \text{ GeV}, \quad f_{B_s} = (0.2303 \pm 0.0013) \text{ GeV}. \quad (\text{D.7})$$

The B -meson masses are taken from the Particle Data Group [96],

$$m_{B^+} = 5.27934 \text{ GeV}, \quad m_{B_d} = 5.27965 \text{ GeV}, \quad m_{B_s} = 5.36688 \text{ GeV}. \quad (\text{D.8})$$

The CKM matrix elements are taken from the global fit of the CKMfitter collaboration [130] (online update). The HQE matrix elements entering the dimension-three, -five, and -six contributions are summarised in Table D.1. The corresponding Bag parameters parameterising the forward matrix elements of the dimension-six $\Delta B = 0$ four-quark operators are collected in Table D.2.²

Parameter	B^+, B_d	Source	B_s	Source
$\mu_\pi^2(B_q) [\text{GeV}^2]$	0.454 ± 0.043	Exp. fit [128]	0.534 ± 0.074	Exp. fit + Ref. [102]
$\mu_G^2(B_q) [\text{GeV}^2]$	0.274 ± 0.053	Exp. fit [128]	0.321 ± 0.072	Exp. fit + Ref. [102]
$\rho_D^3(B_q) [\text{GeV}^3]$	0.176 ± 0.019	Exp. fit [128]	0.210 ± 0.034	Exp. fit + Ref. [102]

Table D.1: Input values of the two-quark non-perturbative parameters.

$\mu_0 = 1.5 \text{ GeV}$	$\tilde{B}_1^q$	$\tilde{B}_2^q$	$\tilde{B}_3^q$	$\tilde{B}_4^q$
$\langle B_{u,d} \tilde{O}_i^{u,d} B_{u,d} \rangle$	$1.0026^{+0.0246}_{-0.0221}$	$0.9982^{+0.0206}_{-0.0214}$	$-0.0057^{+0.0221}_{-0.0225}$	$-0.0014^{+0.0216}_{-0.0221}$
$\langle B_s \tilde{O}_i^s B_s \rangle$	$1.0022^{+0.0246}_{-0.0221}$	$0.9983^{+0.0246}_{-0.0221}$	$-0.0036^{+0.0265}_{-0.0270}$	$-0.0009^{+0.0259}_{-0.0265}$

Table D.2: Input values of the dimension-six Bag parameters at the renormalisation scale $\mu_0 = 1.5 \text{ GeV}$, collected from Ref. [102] and references therein.

Tables D.3 and D.4 define the effective B -Mesogenesis couplings $G_X^{f_1 f_2}$ and list their corresponding upper bounds from Ref. [31]. Here, $X = \text{I}, \dots, \text{VI}$ denotes the topologies shown in Figure 4.1, whereas f_1 and f_2 identify the internal fermions propagating in the loop of the corresponding diagram.

²The parametrisations of the SM and Mesogenesis four-quark operators are identical, since the two operator bases differ only by their chiral structure; see Ref. [131].

Table D.3: $\mathcal{H}_{-1/3}$ Constants and their Max values, taken from Ref. [31].

Parameter	Max Value (GeV) ⁻⁴	Source
$G_{III}^{\psi u} = \frac{y_{ub}y_{\psi d}}{M_Y^2} \frac{y_{ub}^*y_{\psi d}^*}{M_Y^2}$	1×10^{-13}	Table II
$G_{III}^{\psi c} = \frac{y_{cb}y_{\psi d}}{M_Y^2} \frac{y_{cb}^*y_{\psi d}^*}{M_Y^2}$	1.5×10^{-12}	Table II
$G_{III}^{u\psi} = \frac{y_{ud}y_{\psi b}}{M_Y^2} \frac{y_{ud}^*y_{\psi b}^*}{M_Y^2}$	4.7×10^{-15}	Table II
$G_{III}^{c\psi} = \frac{y_{cd}y_{\psi b}}{M_Y^2} \frac{y_{cd}^*y_{\psi b}^*}{M_Y^2}$	1.6×10^{-13}	Table II
$G_{II}^{\psi u} = \frac{y_{ub}y_{\psi d}}{M_Y^2} \frac{y_{ud}^*y_{\psi b}^*}{M_Y^2}$	2.2×10^{-14}	Table II
$G_{II}^{\psi c} = \frac{y_{cb}y_{\psi d}}{M_Y^2} \frac{y_{cd}^*y_{\psi b}^*}{M_Y^2}$	4.9×10^{-13}	Table II
$G_{II}^{u\psi} = \frac{y_{ud}y_{\psi b}}{M_Y^2} \frac{y_{ub}^*y_{\psi d}^*}{M_Y^2}$	2.2×10^{-14}	Table II
$G_{II}^{c\psi} = \frac{y_{cd}y_{\psi b}}{M_Y^2} \frac{y_{cb}^*y_{\psi d}^*}{M_Y^2}$	4.9×10^{-13}	Table II
$G_I^{\psi\psi} = \frac{y_{\psi d}y_{\psi b}^*}{M_Y^2} \frac{y_{\psi b}y_{\psi d}^*}{M_Y^2}$	1.7×10^{-15}	Eq. (43j), Eq. (41a)
$G_I^{uu} = \frac{y_{ud}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{ud}^*}{M_Y^2}$	1.7×10^{-15}	Eq. (43f), Eq. (41a)
$G_I^{cc} = \frac{y_{cd}y_{cb}^*}{M_Y^2} \frac{y_{cb}y_{cd}^*}{M_Y^2}$	1.7×10^{-15}	Eq. (43f), Eq. (41a)
$G_I^{cu} = \frac{y_{ud}y_{cb}^*}{M_Y^2} \frac{y_{cb}y_{ud}^*}{M_Y^2}$	8.5×10^{-16}	Eq. (43e), Eq. (41a)
$G_I^{uc} = \frac{y_{cd}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{cd}^*}{M_Y^2}$	8.5×10^{-16}	Eq. (43e), Eq. (41a)
$G_{IV}^{s\psi} = \frac{y_{\psi s}y_{ub}}{M_Y^2} \frac{y_{ub}^*y_{\psi s}^*}{M_Y^2}$	1×10^{-13}	Table II
$G_{IV}^{d\psi} = \frac{y_{\psi d}y_{ub}}{M_Y^2} \frac{y_{ub}^*y_{\psi d}^*}{M_Y^2}$	1×10^{-13}	Table II
$G_{III}^{\psi s} = \frac{y_{\psi b}y_{us}}{M_Y^2} \frac{y_{us}^*y_{\psi b}^*}{M_Y^2}$	3.7×10^{-14}	Table II
$G_{III}^{\psi d} = \frac{y_{\psi b}y_{ud}}{M_Y^2} \frac{y_{ud}^*y_{\psi b}^*}{M_Y^2}$	4.7×10^{-15}	Table II
$G_V^{\psi s} = \frac{y_{\psi s}y_{ub}}{M_Y^2} \frac{y_{us}^*y_{\psi b}^*}{M_Y^2}$	6.3×10^{-14}	Table II
$G_V^{\psi d} = \frac{y_{\psi d}y_{ub}}{M_Y^2} \frac{y_{ud}^*y_{\psi b}^*}{M_Y^2}$	2.2×10^{-14}	Table II
$G_{VI}^{\psi s} = \frac{y_{\psi b}y_{us}}{M_Y^2} \frac{y_{ub}^*y_{\psi s}^*}{M_Y^2}$	6.3×10^{-14}	Table II
$G_{VI}^{\psi d} = \frac{y_{\psi b}y_{ud}}{M_Y^2} \frac{y_{ub}^*y_{\psi d}^*}{M_Y^2}$	2.2×10^{-14}	Table II
Continued on the next page		

Table D.3 – continued from previous page

Parameter	Max Value (GeV) ⁻⁴	Source
$G_{IV}^{cs} = \frac{y_{cs}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{cs}^*}{M_Y^2}$	8.5×10^{-15}	Eq. (43g), Eq. (41a)
$G_{IV}^{us} = \frac{y_{us}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{us}^*}{M_Y^2}$	8.5×10^{-15}	Eq. (43g), Eq. (41a)
$G_{IV}^{cd} = \frac{y_{cd}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{cd}^*}{M_Y^2}$	8.5×10^{-16}	Eq. (43e), Eq. (41a)
$G_{IV}^{ud} = \frac{y_{ud}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{ud}^*}{M_Y^2}$	1.7×10^{-15}	Eq. (43f), Eq. (41a)
$G_I^{cs} = \frac{y_{us}y_{cb}^*}{M_Y^2} \frac{y_{cb}y_{us}^*}{M_Y^2}$	8.5×10^{-15}	Eq. (43g), Eq. (41a)
$G_I^{us} = \frac{y_{us}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{us}^*}{M_Y^2}$	8.5×10^{-15}	Eq. (43g), Eq. (41a)
$G_I^{cd} = \frac{y_{ud}y_{cb}^*}{M_Y^2} \frac{y_{cb}y_{ud}^*}{M_Y^2}$	8.5×10^{-16}	Eq. (43e), Eq. (41a)
$G_I^{ud} = \frac{y_{ud}y_{ub}^*}{M_Y^2} \frac{y_{ub}y_{ud}^*}{M_Y^2}$	1.7×10^{-15}	Eq. (43f), Eq. (41a)

Table D.4: $\mathcal{H}_{2/3}$ Constants and their Max values, taken from Ref. [31].

Parameter	Max Value (GeV) ⁻⁴	Source
$G_{IV}^{u\psi} = \frac{y_{bd}y_{\psi u}}{M_Y^2} \frac{y_{bd}^*y_{\psi u}^*}{M_Y^2}$	2.5×10^{-13}	Table II
$G_{IV}^{c\psi} = \frac{y_{bd}y_{\psi c}}{M_Y^2} \frac{y_{bd}^*y_{\psi c}^*}{M_Y^2}$	2.5×10^{-13}	Table II
$G_{IV}^{ds} = \frac{y_{ds}y_{bd}^*}{M_Y^2} \frac{y_{bd}y_{ds}^*}{M_Y^2}$	8.5×10^{-15}	Eq. (44d), Eq. (41a)
$G_I^{ds} = \frac{y_{ds}y_{bd}^*}{M_Y^2} \frac{y_{bd}y_{ds}^*}{M_Y^2}$	8.5×10^{-15}	Eq. (44d), Eq. (41a)
$G_I^{ss} = \frac{y_{ds}y_{bs}^*}{M_Y^2} \frac{y_{bs}y_{ds}^*}{M_Y^2}$	1.7×10^{-15}	Eq. (44c), Eq. (41a)
$G_{III}^{d\psi} = \frac{y_{bd}y_{\psi u}}{M_Y^2} \frac{y_{bd}^*y_{\psi u}^*}{M_Y^2}$	2.5×10^{-13}	Table II
$G_{III}^{s\psi} = \frac{y_{bs}y_{\psi u}}{M_Y^2} \frac{y_{bs}^*y_{\psi u}^*}{M_Y^2}$	1.2×10^{-12}	Table II

D.2 Power-Suppressed Effects in the SM: A Comparison

In Section 4.2.1, we showed that within the Mesogenesis framework, power-suppressed contributions can exceed the leading dimension-three term in the HQE in certain

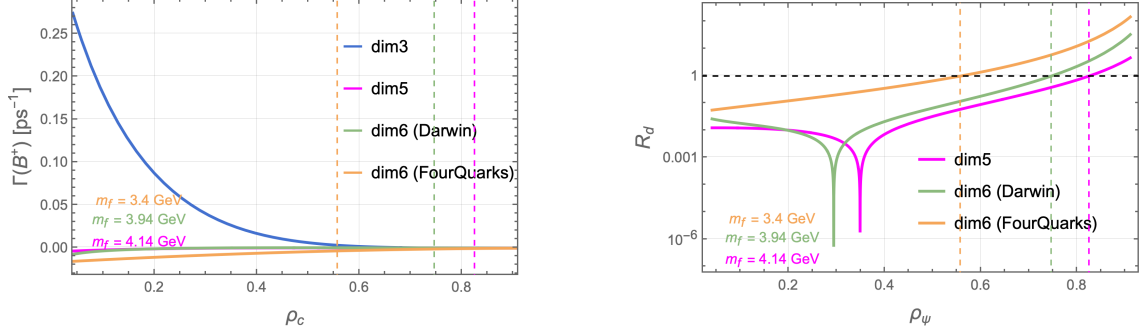


Figure D.1: Total decay width $\Gamma(B^+)$ arising from the SM operator $(cb)(du)$, shown as a function of $\rho_f = (m_f/m_b)^2$. The plots display the individual HQE contributions (left) and the corresponding ratios $R_d = |\Gamma_d/\Gamma_3|$ (right). The vertical lines indicate the values of ρ_f where subleading terms exceed the leading contribution (the corresponding m_f values are also indicated). For the dimension-six four-quark contribution, which originates from the PI topology and is negative across the full range, the line marks the point where its *absolute value* exceeds Γ_3 . Both C_1 and C_2 are included. The sharp spikes in the R_d plot occur where the numerator, Γ_d , changes sign.

regions of the dark-sector fermion mass m_ψ . It is instructive to compare this behaviour with the corresponding situation in the SM. Since the Mesogenesis analysis is performed under the assumption that a single operator dominates at a time, we consider the analogous SM operator $\mathcal{O} = (ud)(cb)$, which induces the familiar PI topology in the B^+ meson. We therefore study the HQE contributions to the inclusive decay $b \rightarrow c\bar{u}d$ as a function of the scaled charm-quark mass $\rho_c = m_c^2/m_b^2$. The corresponding SM effective Hamiltonian reads

$$\mathcal{H}_{\text{eff}}^{\text{SM}} = \frac{4G_F}{\sqrt{2}} V_{cb}V_{ud}^* [C_1(\mu_b) Q_1^{cud} + C_2(\mu_b) Q_2^{cud}] + \text{h.c.}, \quad (\text{D.9})$$

where $Q_{1,2}^{cud}$ are the current–current operators defined in Eq. (1.9).

Using the LO Wilson coefficients $C_1(4.5 \text{ GeV}) = 1.11$ and $C_2(4.5 \text{ GeV}) = -0.26$ from Ref. [102], together with the numerical inputs collected in Section D.1, we show in Figure D.1 the individual HQE contributions and their ratios to the leading dimension-three term. By varying the charm-quark mass over the kinematic range analogous to that spanned by the dark-fermion mass m_ψ in the Mesogenesis analysis, we observe a qualitatively similar hierarchy: in certain regions of parameter space, power-suppressed contributions become larger than the leading term. This demonstrates that the breakdown of the HQE hierarchy observed in the Mesogenesis scenario is not a peculiarity of that model, but rather a generic feature of the HQE when the relevant mass scales approach the kinematic boundaries of the decay.

Despite this qualitative similarity, the size of the power corrections differs between the two scenarios. In the SM, the dimension-six four-quark contributions generated by PI topology remain modest compared with the leading dimension-three term throughout the physical parameter space. In the Mesogenesis framework, however, the corresponding $\overline{\text{PI}}$ topology can produce much larger corrections, causing the

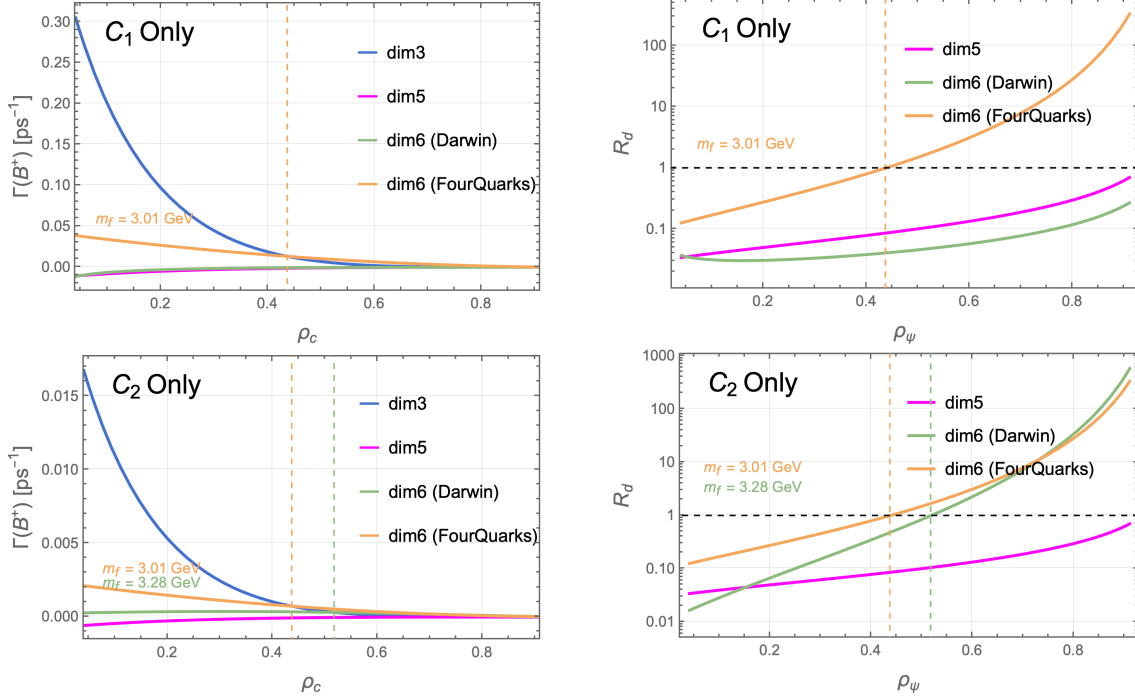


Figure D.2: Total decay width $\Gamma(B^+)$ arising from the SM operator $(cb)(du)$, as a function of $\rho_f = (m_f/m_b)^2$. The plots show the individual HQE contributions (left) and the ratios $R_d = |\Gamma_d/\Gamma_3|$ (right). The upper panels display the case with only C_1 switched on, while the lower panels correspond to only C_2 . Vertical lines mark the values of ρ_f where subleading terms exceed the leading dimension-three width (the corresponding m_f values are also indicated).

power-suppressed terms to compete with—or even exceed—the leading contribution, as demonstrated in Figures 4.2 and 4.3.

To isolate the differences between the SM PI topology and the corresponding $\overline{\text{PI}}$ topology in Mesogenesis, we first eliminate the interference between the two current–current operators by setting $C_2 = 0$. The SM Hamiltonian then reduces to

$$\mathcal{H}_{b \rightarrow cud}^{\text{SM}}(x) = \tilde{G}_F (\bar{c}^i \gamma_\mu P_L b^i) (\bar{d}^j \gamma^\mu P_L u^j)(x) + \text{h.c.}, \quad (\text{D.10})$$

where $\tilde{G}_F = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* C_1$. The corresponding Mesogenesis operator is

$$\mathcal{H}_{b \rightarrow \psi ud}^{-1/3}(x) = C_\psi^{(ud)} \epsilon_{ijk} (\bar{u}_R^i b_R^{c,j}) (\bar{\psi} d_R^{c,k})(x) + \text{h.c.}. \quad (\text{D.11})$$

The two operators differ only in their Dirac and colour structures. These differences can be identified separately already at leading order. Replacing $|C_\psi^{(ud)}|^2 \rightarrow |\tilde{G}_F|^2$, the SM dimension-three contribution is related to the corresponding Mesogenesis result by

$$\Gamma_3^{\text{SM}}(B^+) = \Gamma_3^{-1/3}(B^+) \times \underbrace{4}_{\text{Dirac}} \times \underbrace{\frac{3}{2}}_{\text{color}}. \quad (\text{D.12})$$

The same Dirac and colour structures also determine the dimension-six four-quark

contributions. In the SM, the PI contribution is

$$\Gamma_{\text{PI}}^{\text{SM}} \propto 4 \left(\frac{1}{N_c} \tilde{Q}_1 + 2\tilde{Q}_3 \right), \quad (\text{D.13})$$

where the operators $\tilde{Q}_{1,3}$ are defined in Eq. (D.3). By contrast, the corresponding $\overline{\text{PI}}$ contribution in Mesogenesis reads

$$\Gamma_{\overline{\text{PI}}}^{-1/3} \propto \left(\frac{N_c - 1}{N_c} \tilde{Q}_1 - 2\tilde{Q}_3 \right) \Big|_{P_L \rightarrow P_R}. \quad (\text{D.14})$$

The SM contribution retains the overall factor of 4 originating from the Dirac structure. The colour structure, however, differs from that of the Mesogenesis operator, leading to a different linear combination of the four-quark operators.

Employing the parametrisation of the forward matrix elements in terms of the bag parameters listed in Table D.2, we find that the combination $\left(\frac{N_c - 1}{N_c} \right) \tilde{B}_1^u - 2\tilde{B}_3^u$ appearing in Mesogenesis is approximately twice as large in magnitude as the corresponding SM combination $\left(\frac{1}{N_c} \right) \tilde{B}_1^u + 2\tilde{B}_3^u$. This modified colour structure therefore enhances the dimension-six four-quark contribution in Mesogenesis relative to the SM.

Finally, reinstating the full SM operator basis by including the interference with the operator proportional to C_2 further suppresses the dimension-six four-quark contribution. Numerically, the magnitude of the PI term is reduced by approximately a factor of two, shifting the point at which it overtakes the leading dimension-three contribution from $\rho_c \simeq 0.4$ (for $C_2 = 0$) to $\rho_c \simeq 0.6$ (for the physical values of C_1 and C_2), as shown in Figure D.2. No analogous cancellation occurs in the Mesogenesis framework, where a single operator is assumed to dominate, allowing the enhanced bag-parameter combination to remain fully effective.

Taken together, these observations explain why the relative size of the $\overline{\text{PI}}$ contribution is substantially larger in Mesogenesis than the corresponding PI contribution in the SM. Two effects are responsible for this enhancement. First, the different colour structure of the Mesogenesis operator selects a more favorable combination of bag parameters. Second, under the assumption of single-operator dominance, the Mesogenesis contribution is not subject to the destructive interference between current-current operators that further suppresses the PI contribution in the SM. Therefore, the ratio $\Gamma_{\text{dim6}}/\Gamma_{\text{dim3}}$ is systematically enhanced in the Mesogenesis framework, allowing the dimension-six contribution to exceed the leading dimension-three term over sizable regions of parameter space. For calculations involving BSM operators with the same colour structure as the SM but different Dirac structures, see Ref. [81].

D.3 B -meson Distribution Amplitudes

D.3.1 Two-Particle B -meson DAs

The nonlocal vacuum-to- B matrix element appearing in Eq. (4.69) is parameterised in terms of the two-particle B -meson light-cone distribution amplitudes (DAs).

Employing the HQET decomposition of the heavy-quark field, we use the standard parametrisation [114] (for its derivation, see Section D.3.3) ³

$$i\langle 0|\bar{h}_{v\alpha}(0)[0,x]u_\beta(x)|\bar{B}(v)\rangle = \frac{f_B m_B}{4} \int_0^\infty d\omega e^{-i\omega v \cdot x} \left[\left\{ \phi_+^B(\omega) + x^2 g_+^B(\omega) \right\} \left[(1 + \not{v}) \gamma_5 \right]_{\beta\alpha} \right. \\ \left. - \frac{1}{2v \cdot x} \left\{ \phi_+^B(\omega) - \phi_-^B(\omega) + x^2 (g_+^B(\omega) - g_-^B(\omega)) \right\} \left[\not{x} (1 + \not{v}) \gamma_5 \right]_{\beta\alpha} \right], \quad (\text{D.15})$$

where $[0, x]$ denotes the gauge link ensuring gauge invariance, and ω corresponds to the light-cone projection of the spectator quark momentum in the B -meson rest frame.

The functions $\phi_+^B(\omega)$, $\phi_-^B(\omega)$, $g_+^B(\omega)$, and $g_-^B(\omega)$ represent the two-particle B -meson DAs of twist two, three, four, and five, respectively, following the classification of [133]. The twist-two and twist-three DAs satisfy the normalisation conditions

$$\int_0^\infty d\omega \phi_\pm^B(\omega) = 1. \quad (\text{D.16})$$

To eliminate the explicit factors $1/(v \cdot x)$ appearing in Eq. (D.15), it is convenient to introduce the integrated combination [134]

$$\Phi_\pm^B(\omega) \equiv \int_0^\omega d\tau (\phi_\pm^B(\tau) - \phi_\mp^B(\tau)), \quad (\text{D.17})$$

such that

$$\frac{d}{d\omega} \Phi_\pm^B(\omega) = \phi_\pm^B(\omega) - \phi_\mp^B(\omega), \quad \Phi_\pm^B(0) = \Phi_\pm^B(\infty) = 0.$$

Analogously, we define

$$G_\pm^B(\omega) \equiv \int_0^\omega d\tau (g_\pm^B(\tau) - g_\mp^B(\tau)). \quad (\text{D.18})$$

For the numerical analysis we adopt the standard exponential model for the twist-two DA [114],

$$\phi_+^B(\omega) = \frac{\omega}{\lambda_B^2} e^{-\omega/\lambda_B}, \quad (\text{D.19})$$

while for the twist-three and twist-four DAs we employ the corresponding exponential ansatz proposed in [133]

$$\phi_-^B(\omega) = \frac{1}{\lambda_B} e^{-\omega/\lambda_B} + \frac{1}{2\lambda_B} \left[1 - \frac{2\omega}{\lambda_B} + \frac{\omega^2}{2\lambda_B^2} \right] \frac{1-R}{1+2R} e^{-\omega/\lambda_B}, \quad (\text{D.20})$$

³The parametrisation given here differs by an overall minus sign from that used in Ref. [132]. This originates from a different convention for the charge-conjugation matrix, where Ref. [132] adopts $C = \gamma^2 \gamma^0$. While such an overall phase does not affect physical observables, since the form factors enter quadratically in the branching fraction (see Eq. (4.95)), the same convention also enters the perturbative derivation of the correlation function in Eq. (4.44) through the charge-conjugation identities. Consequently, the overall sign is compensated, and the final results are identical.

$$g_+^B(\omega) = \frac{3\omega^2}{16\lambda_B} \frac{3+4R}{1+2R} e^{-\omega/\lambda_B}. \quad (\text{D.21})$$

For the twist-five DA we employ the Wandzura–Wilczek inspired model used in [135],

$$g_-^B(\omega) = \frac{3\omega}{4} e^{-\omega/\lambda_B}. \quad (\text{D.22})$$

To assess the numerical impact of the twist-five contribution, we also consider the approximation in which the term proportional to g_-^B is neglected. In this case, one simply replaces

$$G_\pm^B(\omega) \rightarrow G_+^B(\omega) = \int_0^\omega d\tau g_+^B(\tau) \quad (\text{D.23})$$

in the final expressions for the invariant amplitudes.

The set of B -meson DAs employed in this work depends on three nonperturbative parameters: the decay constant f_B , the inverse moment of the twist-two DA,

$$\frac{1}{\lambda_B} = \int_0^\infty d\omega \frac{\phi_+^B(\omega)}{\omega}, \quad (\text{D.24})$$

and the ratio

$$R \equiv \frac{\lambda_E^2}{\lambda_H^2}, \quad (\text{D.25})$$

where λ_E and λ_H parameterize the matrix elements of local heavy-light quark-antiquark-gluon operators in HQET.

D.3.2 Three-Particle B -meson DAs

In addition to the two-particle contributions discussed above, the OPE also receives corrections from three-particle quark-antiquark-gluon B -meson DAs. For completeness, we summarise here the corresponding definitions and conventions used in the present work.

We employ the HQET parametrisation of the vacuum-to- B matrix element with one gluon-field insertion derived in [133]. More specifically, we use the form presented in [136] for the $\bar{B}$ meson, rewritten with explicit Dirac indices and charge-conjugated to the B^+ matrix element, adopting the same charge-conjugation convention as in the two-particle DA parametrisation in Eq. (D.15):

$$\begin{aligned} \langle 0 | \bar{h}_{v\alpha}(0) g_s \frac{\lambda^a}{2} G_{\mu\nu}^a(ux) u_\beta(x) | B^+(v) \rangle &= \frac{f_B m_B}{4} \int_0^\infty d\omega_1 d\omega_2 e^{-i(\omega_1 + u\omega_2)v \cdot x} \\ &\times \left\{ \gamma_5 \left[- (v_\mu \gamma_\nu - v_\nu \gamma_\mu) \phi_3 - \frac{i}{2} \sigma_{\mu\nu} (\phi_3 - \phi_4) + \frac{x_\mu v_\nu - x_\nu v_\mu}{2v \cdot x} (\phi_3 + \phi_4 - 2\psi_4) \right. \right. \\ &\quad + \frac{x_\mu \gamma_\nu - x_\nu \gamma_\mu}{2v \cdot x} (\phi_3 + \tilde{\psi}_4) + \frac{i\epsilon_{\mu\nu\alpha\beta} x^\alpha v^\beta}{2v \cdot x} \gamma_5 (\phi_3 - \phi_4 + 2\tilde{\psi}_4) \\ &\quad - \frac{i\epsilon_{\mu\nu\alpha\beta} x^\alpha}{2v \cdot x} \gamma^\beta \gamma_5 (\phi_3 - \phi_4 + \tilde{\psi}_4) + \frac{(x_\mu v_\nu - x_\nu v_\mu) \not{x}}{2(v \cdot x)^2} (\phi_4 - \psi_4 - \tilde{\psi}_4) \\ &\quad \left. \left. - \frac{\not{x} (x_\mu \gamma_\nu - x_\nu \gamma_\mu)}{4(v \cdot x)^2} (\phi_3 - \phi_4 + 2\tilde{\psi}_4) \right] (1 - \not{v}) \right\}_{\beta\alpha}. \quad (\text{D.26}) \end{aligned}$$

Here ϕ_3 denotes the twist-three DA, while ϕ_4 , ψ_4 , and $\tilde{\psi}_4$ are twist-four DAs. For brevity, their dependence on (ω_1, ω_2) is suppressed, and the Wilson lines on the left-hand side are not shown explicitly.

Similarly to the two-particle case, it is convenient to eliminate the explicit factors of $1/(v \cdot x)$ by introducing integrated DAs:

$$\bar{\Phi}(\omega_1, \omega_2) \equiv \int_0^{\omega_1} d\tau \Phi(\tau, \omega_2), \quad (\text{D.27})$$

where

$$\Phi = \{\phi_3, \phi_4, \psi_4, \tilde{\psi}_4\}.$$

To remove the terms proportional to $1/(v \cdot x)^2$, we further define the doubly integrated DAs

$$\bar{\bar{\Phi}}(\omega_1, \omega_2) \equiv \int_0^{\omega_1} d\tau \bar{\Phi}(\tau, \omega_2). \quad (\text{D.28})$$

Using these definitions, the matrix element [Eq. \(D.26\)](#) can be rewritten in a more compact form free of inverse powers of $(v \cdot x)$:

$$\begin{aligned} \langle 0 | \bar{h}_{v\alpha}(0) g_s \frac{\lambda^a}{2} G_{\mu\nu}^a(ux) u_\beta(x) | B^+(v) \rangle &= -\frac{f_B m_B}{4} \int_0^\infty d\omega_1 d\omega_2 e^{-i\sigma m_B v \cdot x} \\ &\times \left\{ \gamma_5 \left[- (v_\mu \gamma_\nu - v_\nu \gamma_\mu) \phi_3 - \frac{i}{2} \sigma_{\mu\nu} (\phi_3 - \phi_4) + \frac{i}{2} (x_\mu v_\nu - x_\nu v_\mu) (\bar{\phi}_3 + \bar{\phi}_4 - 2\bar{\psi}_4) \right. \right. \\ &\quad + \frac{i}{2} (x_\mu \gamma_\nu - x_\nu \gamma_\mu) (\bar{\phi}_3 + \bar{\tilde{\psi}}_4) - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} x^\alpha v^\beta \gamma_5 (\bar{\phi}_3 - \bar{\phi}_4 + 2\bar{\tilde{\psi}}_4) \\ &\quad + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} x^\alpha \gamma^\beta \gamma_5 (\bar{\phi}_3 - \bar{\phi}_4 + \bar{\tilde{\psi}}_4) - \frac{1}{2} (x_\mu v_\nu - x_\nu v_\mu) \not{x} (\bar{\bar{\phi}}_4 - \bar{\bar{\psi}}_4 - \bar{\bar{\tilde{\psi}}}_4) \\ &\quad \left. \left. + \frac{1}{4} \not{x} (x_\mu \gamma_\nu - x_\nu \gamma_\mu) (\bar{\bar{\phi}}_3 - \bar{\bar{\phi}}_4 + 2\bar{\bar{\tilde{\psi}}}_4) \right] (1 - \not{v}) \right\}_{\beta\alpha}. \end{aligned} \quad (\text{D.29})$$

where we introduced

$$\sigma(u, \omega_1, \omega_2) \equiv \frac{\omega_1 + u\omega_2}{m_B}. \quad (\text{D.30})$$

For completeness, we also quote the exponential model for the three-particle DAs proposed in [\[133\]](#):

$$\begin{aligned} \phi_3(\omega_1, \omega_2) &= -\frac{3\omega_1\omega_2^2}{4\lambda_B^3} \frac{1-R}{1+2R} e^{-(\omega_1+\omega_2)/\lambda_B}, \\ \phi_4(\omega_1, \omega_2) &= \frac{3\omega_2^2}{4\lambda_B^2} \frac{1+R}{1+2R} e^{-(\omega_1+\omega_2)/\lambda_B}, \\ \psi_4(\omega_1, \omega_2) &= R \tilde{\psi}_4(\omega_1, \omega_2) = \frac{3\omega_1\omega_2}{2\lambda_B^2} \frac{R}{1+2R} e^{-(\omega_1+\omega_2)/\lambda_B}. \end{aligned} \quad (\text{D.31})$$

All three-particle DAs in this model are therefore determined by the same nonperturbative parameters λ_B and R already entering the two-particle DAs.

D.3.3 B^+ -meson DAs from the B^- -meson Parametrisation

In this section, we obtain the parametrisation of the B^+ -meson light-cone DAs from the corresponding B^- -meson parametrisation using charge-conjugation symmetry.

We start from the parametrisation in Ref. [137]

$$\begin{aligned} \langle 0 | \bar{q}_\beta(z) b_\alpha(0) | B^-(v) \rangle &= -\frac{if_B M_B}{4} \int_0^\infty d\omega e^{-i\omega v \cdot z} \\ &\times \left[(1 + \not{v}) \left(\phi_+^B(\omega) - \frac{\phi_+^B(\omega) - \phi_-^B(\omega)}{2v \cdot z} \not{z} \right) \gamma_5 \right]_{\alpha\beta}. \end{aligned} \quad (\text{D.32})$$

For simplicity, the gauge link between the quark fields is suppressed throughout this derivation. We write this in the form

$$\langle 0 | \bar{q}_\beta(z) b_\alpha(0) | B^-(v) \rangle = A [(1 + \not{v}) \gamma_5]_{\alpha\beta} + B [(1 + \not{v}) \not{z} \gamma_5]_{\alpha\beta}, \quad (\text{D.33})$$

with

$$A = -\frac{if_B M_B}{4} \int_0^\infty d\omega e^{-i\omega v \cdot z} \phi_+^B(\omega), \quad (\text{D.34})$$

$$B = \frac{if_B M_B}{8v \cdot z} \int_0^\infty d\omega e^{-i\omega v \cdot z} (\phi_+^B(\omega) - \phi_-^B(\omega)). \quad (\text{D.35})$$

We now obtain the corresponding B^+ matrix element by applying charge conjugation with the operator $\hat{C}$:

$$\langle 0 | \bar{q}_\beta(z) b_\alpha(0) | B^- \rangle = \langle 0 | \hat{C}^{-1} \underbrace{\hat{C} \bar{q}_\beta(z) \hat{C}^{-1}} \underbrace{\hat{C} b_\alpha(0) \hat{C}^{-1}} \hat{C} | B^- \rangle. \quad (\text{D.36})$$

Using (see the relations in Eq. (A.16))⁴

$$\hat{C} | B^- \rangle = | B^+ \rangle, \quad (\text{D.37})$$

$$\hat{C} \bar{q}_\beta(z) \hat{C}^{-1} = -C_{\beta\beta'} q_{\beta'}(z), \quad (\text{D.38})$$

$$\hat{C} b_\alpha(0) \hat{C}^{-1} = C_{\alpha'\alpha}^{-1} \bar{b}_{\alpha'}(0), \quad (\text{D.39})$$

one obtains

$$\langle 0 | \bar{b}_{\alpha'}(0) q_{\beta'}(z) | B^+ \rangle C_{\alpha'\alpha}^{-1} C_{\beta\beta'} = A [(1 + \not{v}) \gamma_5]_{\alpha\beta} + B [(1 + \not{v}) \not{z} \gamma_5]_{\alpha\beta}. \quad (\text{D.40})$$

Multiplying by $C_{\alpha\alpha_2}$ and $C_{\beta_2\beta}^{-1}$ gives

$$\langle 0 | \bar{b}_{\alpha_2}(0) q_{\beta_2}(z) | B^+ \rangle = A \left[C(1 + \not{v}) \gamma_5 C^{-1} \right]_{\alpha_2\beta_2} + B \left[C(1 + \not{v}) \not{z} \gamma_5 C^{-1} \right]_{\alpha_2\beta_2}. \quad (\text{D.41})$$

$$C(1 + \not{v}) \gamma_5 C^{-1} = (1 - \not{v}^T) \gamma_5^T = [\gamma_5(1 - \not{v})]^T = [(1 + \not{v}) \gamma_5]^T, \quad (\text{D.42})$$

⁴Note that $\hat{C}$ denotes the charge-conjugation *operator*, whereas C denotes the corresponding charge-conjugation *matrix* acting on Dirac indices.

$$\begin{aligned}
C(1 + \not{v})\not{z}\gamma_5 C^{-1} &= (1 - \not{v}^T)(-\not{z}^T)\gamma_5^T = -[\gamma_5 \not{z}(1 - \not{v})]^T \\
&= [\not{z}(1 + \not{v})\gamma_5]^T.
\end{aligned} \tag{D.43}$$

Therefore,

$$\langle 0 | \bar{b}_\alpha(0) q_\beta(z) | B^+ \rangle = A [(1 + \not{v})\gamma_5]_{\beta\alpha} + B [\not{z}(1 + \not{v})\gamma_5]_{\beta\alpha}. \tag{D.44}$$

Finally, substituting the expressions for A and B yields

$$\begin{aligned}
\langle 0 | \bar{b}_\alpha(0) q_\beta(z) | B^+(v) \rangle &= -\frac{if_B M_B}{4} \int_0^\infty d\omega e^{-i\omega v \cdot z} \\
&\times \left[\phi_+^B(\omega) (1 + \not{v})\gamma_5 - \frac{\phi_+^B(\omega) - \phi_-^B(\omega)}{2v \cdot z} \not{z}(1 + \not{v})\gamma_5 \right]_{\beta\alpha},
\end{aligned} \tag{D.45}$$

which coincides with the expression in Ref. [114].

D.4 B -Mesogenesis Additional Results

D.4.1 Inclusive Results from HQE

In this section we collect the explicit expressions for the coefficients entering Eqs. (4.26), (4.33) and (4.34).

Throughout this appendix, we employ the abbreviations

$$\Delta_{\rho_\psi} \equiv \sqrt{1 - 4\rho_\psi}, \quad \Lambda_{\rho_\psi} \equiv \log\left(\frac{1 + \Delta_{\rho_\psi}}{1 - \Delta_{\rho_\psi}}\right). \tag{D.46}$$

We start with the two-quark operators.

For $C_{\psi\psi}^{(d)}$:

$$f_0^{(\psi\psi d)} = \frac{1}{2} \left[\Delta_{\rho_\psi} (1 - 14\rho_\psi - 2\rho_\psi^2 - 12\rho_\psi^3) + 24\rho_\psi^2 (1 - \rho_\psi^2) \Lambda_{\rho_\psi} \right] \tag{D.47}$$

$$f_G^{(\psi\psi d)} = \frac{1}{4} \left[24\rho_\psi^2 (1 - 5\rho_\psi^2) \Lambda_{\rho_\psi} - \Delta_{\rho_\psi} (3 - 10\rho_\psi + 10\rho_\psi^2 + 60\rho_\psi^3) \right] \tag{D.48}$$

$$\begin{aligned}
f_D^{(\psi\psi d)} &= \frac{1}{3} \left[\Delta_{\rho_\psi} (9 + 22\rho_\psi - 34\rho_\psi^2 - 60\rho_\psi^3) + 24\Delta_{\rho_\psi} \log\left(\frac{\rho_\psi}{1 - 4\rho_\psi}\right) \right. \\
&\quad \left. + 24(1 - 2\rho_\psi - \rho_\psi^2 - 2\rho_\psi^3 - 5\rho_\psi^4) \Lambda_{\rho_\psi} \right].
\end{aligned} \tag{D.49}$$

For $C_{cu}^{(d)}$:

$$f_0^{(cud)} = -2 \left[\rho_c^4 - 8\rho_c^3 + 8\rho_c + 12\rho_c^2 \log \rho_c - 1 \right], \tag{D.50}$$

$$f_G^{(cud)} = -5\rho_c^4 + 28\rho_c^3 - 48\rho_c^2 + 20\rho_c - 12(\rho_c - 2)\rho_c \log \rho_c + 5, \quad (D.51)$$

$$f_D^{(cud)} = \frac{4}{3} \left[-5\rho_c^4 + 26\rho_c^3 - 45\rho_c^2 - 10\rho_c - 12(\rho_c - 1)^2(\rho_c + 1) \log(1 - \rho_c) \right. \\ \left. + 6(\rho_c^3 - \rho_c^2 + 5\rho_c + 2) \log \rho_c + 34 \right]. \quad (D.52)$$

For $C_{uc}^{(d)}$:

$$f_0^{(ucd)} = f_0^{(cud)}, \quad (D.53)$$

$$f_G^{(ucd)} = -5\rho_c^4 + 16\rho_c^3 - 16\rho_c - 12\rho_c^2 \log \rho_c + 5, \quad (D.54)$$

$$f_D^{(ucd)} = -\frac{2}{3} \left[10\rho_c^4 - 15\rho_c^3 - 18\rho_c^2 + 71\rho_c + 12(\rho_c - 1)^2(\rho_c + 5) \log(1 - \rho_c) \right. \\ \left. - 6(\rho_c^3 + 3\rho_c^2 + 1) \log \rho_c - 48 \right] \quad (D.55)$$

For $C_{cc}^{(d)}$:

$$f_0^{(ccd)} = \frac{1}{4} f_0^{(\psi\psi d)}|_{\rho_\psi \rightarrow \rho_c}, \quad (D.56)$$

$$f_G^{(ccd)} = \Delta_{\rho_c} (5 + 14\rho_c + 14\rho_c^2 - 60\rho_c^3) - 24\rho_c (1 - \rho_c - 2\rho_c^2 + 5\rho_c^3) \Lambda_{\rho_c}, \quad (D.57)$$

$$f_D^{(ccd)} = \frac{2}{3} \left[\Delta_{\rho_c} \left(72 - \rho_c + 22\rho_c^2 - 120\rho_c^3 + 12 \log \frac{\rho_c}{1 - 4\rho_c} \right) \right. \\ \left. - 6 (40\rho_c^4 - 14\rho_c^3 + 8\rho_c^2 + 5\rho_c + 3) \Lambda_{\rho_c} \right]. \quad (D.58)$$

For operators involving two distinct massive final states (e.g. c and ψ), the Darwin term requires two-scale master integrals that are not available in closed form. Since these contributions are expected to be numerically subleading, with the dominant subleading effects arising from the dimension-6 four-quark operators, we restrict the analysis in such cases to the dimension-5 level. The complete expressions for the dimension-3 and dimension-5 terms are lengthy and can be provided upon request.

The above results also apply to the Wilson coefficients obtained by replacing $d \rightarrow s$. For operators with only light quarks (u, d, s) in the final state, the corresponding expressions are obtained in the massless limit of the massive cases.

We now turn to the four-quark contributions. The result for topology IV in model (d), corresponding to the coefficients $\Gamma_{IV,a}^{(\psi ud),u}$, was already derived in [Eq. \(4.24\)](#); for completeness, we repeat it here in the unified notation.

Contributions to $\Gamma(B^+)$:

$$\tilde{\Gamma}_{IV,1}^{(\psi ud),u} = -2(1 - \rho_\psi)^2, \quad \tilde{\Gamma}_{IV,2}^{(\psi ud),u} = \tilde{\Gamma}_{IV,4}^{(\psi ud),u} = 0, \quad \tilde{\Gamma}_{IV,3}^{(\psi ud),u} = 6(1 - \rho_\psi)^2, \quad (D.59)$$

$$\tilde{\Gamma}_{IV,1}^{(\tilde{\psi} ud),u} = -\frac{1}{3}(1 - \rho_\psi)^2(2 + \rho_\psi), \quad \tilde{\Gamma}_{IV,2}^{(\tilde{\psi} ud),u} = \frac{2}{3}(1 - \rho_\psi)^2(2\rho_\psi + 1), \quad (D.60)$$

$$\tilde{\Gamma}_{IV,3}^{(\tilde{\psi}ud),u} = (1 - \rho_\psi)^2(2 + \rho_\psi), \quad \tilde{\Gamma}_{IV,4}^{(\tilde{\psi}ud),u} = -2(1 - \rho_\psi)^2(2\rho_\psi + 1). \quad (D.61)$$

$$\begin{aligned} \tilde{\Gamma}_{I,1}^{(cud),u} &= -\frac{2}{3}(1 - \rho_c)^2(2 + \rho_c), & \tilde{\Gamma}_{I,2}^{(cud),u} &= \frac{4}{3}(1 - \rho_c)^2(2\rho_c + 1), \\ \tilde{\Gamma}_{I,3}^{(cud),u} &= -(1 - \rho_c)^2(2 + \rho_c), & \tilde{\Gamma}_{I,4}^{(cud),u} &= 2(1 - \rho_c)^2(2\rho_c + 1), \end{aligned} \quad (D.62)$$

$$\tilde{\Gamma}_{IV,1}^{(ucd),u} = -4(1 - \rho_c)^2, \quad \tilde{\Gamma}_{IV,2}^{(ucd),u} = \tilde{\Gamma}_{IV,4}^{(ucd),u} = 0, \quad \tilde{\Gamma}_{IV,3}^{(ucd),u} = 12(1 - \rho_c)^2, \quad (D.63)$$

where $\rho_f = m_f^2/m_b^2$. Finally, the massless limit follows from

$$\tilde{\Gamma}_{IV,a}^{(uud),u} = \tilde{\Gamma}_{IV,a}^{(ucd),u} \Big|_{\rho_c \rightarrow 0}, \quad \tilde{\Gamma}_{I,a}^{(uud),u} = \tilde{\Gamma}_{I,a}^{(ucd),u} \Big|_{\rho_c \rightarrow 0}. \quad (D.64)$$

Contributions to $\Gamma(B_q)$:

$$\begin{aligned} \tilde{\Gamma}_{III,1}^{(\psi cd),q} &= \frac{1}{3} \sqrt{\lambda(1, \rho_c, \rho_\psi)} (\rho_\psi^2 + \rho_c^2 + \rho_\psi + \rho_c - 2\rho_\psi\rho_c - 2), \\ \tilde{\Gamma}_{III,2}^{(\psi cd),q} &= \frac{-2}{3} \sqrt{\lambda(1, \rho_c, \rho_\psi)} (2\rho_\psi^2 + 2\rho_c^2 - \rho_\psi - \rho_c - 4\rho_\psi\rho_c - 1), \\ \tilde{\Gamma}_{III,3}^{(\psi cd),q} &= -3\tilde{\Gamma}_{III,1}^{(\psi cd),q}, & \tilde{\Gamma}_{III,4}^{(\psi cd),q} &= -3\tilde{\Gamma}_{III,2}^{(\psi cd),q}, \end{aligned} \quad (D.65)$$

$$\begin{aligned} \tilde{\Gamma}_{I,1}^{(\psi\psi d),q} &= -\sqrt{1 - 4\rho_\psi} (1 - \rho_\psi), & \tilde{\Gamma}_{I,2}^{(\psi\psi d),q} &= \sqrt{1 - 4\rho_\psi} (2\rho_\psi + 1), \\ \tilde{\Gamma}_{I,3}^{(\psi\psi d),q} &= \tilde{\Gamma}_{I,4}^{(\psi\psi d),q} = 0, \end{aligned} \quad (D.66)$$

$$\begin{aligned} \tilde{\Gamma}_{I,1}^{(ccd),q} &= -\frac{4}{3} \sqrt{1 - 4\rho_c} (1 - \rho_c), & \tilde{\Gamma}_{I,2}^{(ccd),q} &= \frac{4}{3} \sqrt{1 - 4\rho_c} (2\rho_c + 1), \\ \tilde{\Gamma}_{I,3}^{(ccd),q} &= -2\sqrt{1 - 4\rho_c} (1 - \rho_c), & \tilde{\Gamma}_{I,4}^{(ccd),q} &= 2\sqrt{1 - 4\rho_c} (2\rho_c + 1), \end{aligned} \quad (D.67)$$

$$\begin{aligned} \tilde{\Gamma}_{I,1}^{(cud),q} &= -\frac{2}{3}(1 - \rho_c)^2(2 + \rho_c), & \tilde{\Gamma}_{I,2}^{(cud),q} &= \frac{4}{3}(1 - \rho_c)^2(2\rho_c + 1), \\ \tilde{\Gamma}_{I,3}^{(cud),q} &= -(1 - \rho_c)^2(2 + \rho_c), & \tilde{\Gamma}_{I,4}^{(cud),q} &= 2(1 - \rho_c)^2(2\rho_c + 1). \end{aligned} \quad (D.68)$$

The following relations hold

$$\begin{aligned} \tilde{\Gamma}_{III,a}^{(\tilde{\psi}cd),q} &= \tilde{\Gamma}_{III,a}^{(\psi cd),q}, & \tilde{\Gamma}_{III,a}^{(\psi ud),q} &= \tilde{\Gamma}_{III,a}^{(\tilde{\psi}ud),q} = \tilde{\Gamma}_{III,a}^{(\psi cd),q} \Big|_{\rho_c \rightarrow 0}, & \tilde{\Gamma}_{I,a}^{(ucd),q} &= \tilde{\Gamma}_{I,a}^{(cud),q}, \\ \tilde{\Gamma}_{I,a}^{(uud),q} &= \tilde{\Gamma}_{I,a}^{(cud),q} \Big|_{\rho_c \rightarrow 0}. \end{aligned} \quad (D.69)$$

Moreover, since in our analysis we work in the massless limit for the light quarks, $m_d = m_s = 0$, the corresponding coefficients above are identical under the interchange $d \leftrightarrow s$.

D.4.2 Exclusive Results from LCSR

In this section we collect the final expressions for the LCSR results derived in Eq. (4.76).

$$\begin{aligned}
\mathcal{C}(\mathcal{P}, v) &= \left[8(\mathcal{P} \cdot v) \not{\mathcal{P}} \not{v} - 16(\mathcal{P} \cdot v)^2 \right] \phi_3 + 8(\mathcal{P} \cdot v - \not{\mathcal{P}} \not{v})(\bar{\phi}_3 - \bar{\phi}_4) \\
&\quad + 8(4\mathcal{P} \cdot v - \not{\mathcal{P}} \not{v}) \bar{\psi}_4 - \frac{4}{\mathcal{P}^2} \left[\mathcal{P}^2 + 2(\mathcal{P} \cdot v) \not{\mathcal{P}} \not{v} \right] (\bar{\phi}_4 - \bar{\psi}_4 - \bar{\bar{\psi}}_4) \\
&\quad - 12(\bar{\phi}_3 - \bar{\phi}_4 + 2\bar{\bar{\psi}}_4), \\
\bar{\mathcal{C}}(\mathcal{P}, v) &= 12\mathcal{P}^2 \text{Log}(-\mathcal{P}^2) \phi_4 - 4(\mathcal{P} \cdot v - \not{\mathcal{P}} \not{v})(\bar{\phi}_3 - \bar{\phi}_4) \\
&\quad - 4(4\mathcal{P} \cdot v - \not{\mathcal{P}} \not{v}) \bar{\psi}_4 + \frac{4}{\mathcal{P}^2} \left[\mathcal{P}^2 + 2(\mathcal{P} \cdot v) \not{\mathcal{P}} \not{v} \right] (\bar{\phi}_4 - \bar{\psi}_4 - \bar{\bar{\psi}}_4) \\
&\quad + 12(\bar{\phi}_3 - \bar{\phi}_4 + 2\bar{\bar{\psi}}_4),
\end{aligned} \tag{D.70}$$

where $\mathcal{P} = P - \sigma m_B v$. In the above expressions, terms that do not generate an imaginary part in the variable P^2 are omitted, since they do not contribute to the final LCSR expressions.

D.4.3 Supplementary Plots

This appendix collects the results for the remaining operators considered in our analysis. While the main text focuses on the representative cases ($\mathcal{O} = (bu)(\psi d)$ and $\mathcal{O} = (du)(\psi b)$), the following plots provide a systematic overview and illustrate to what extent the observed features generalize across the operator basis.

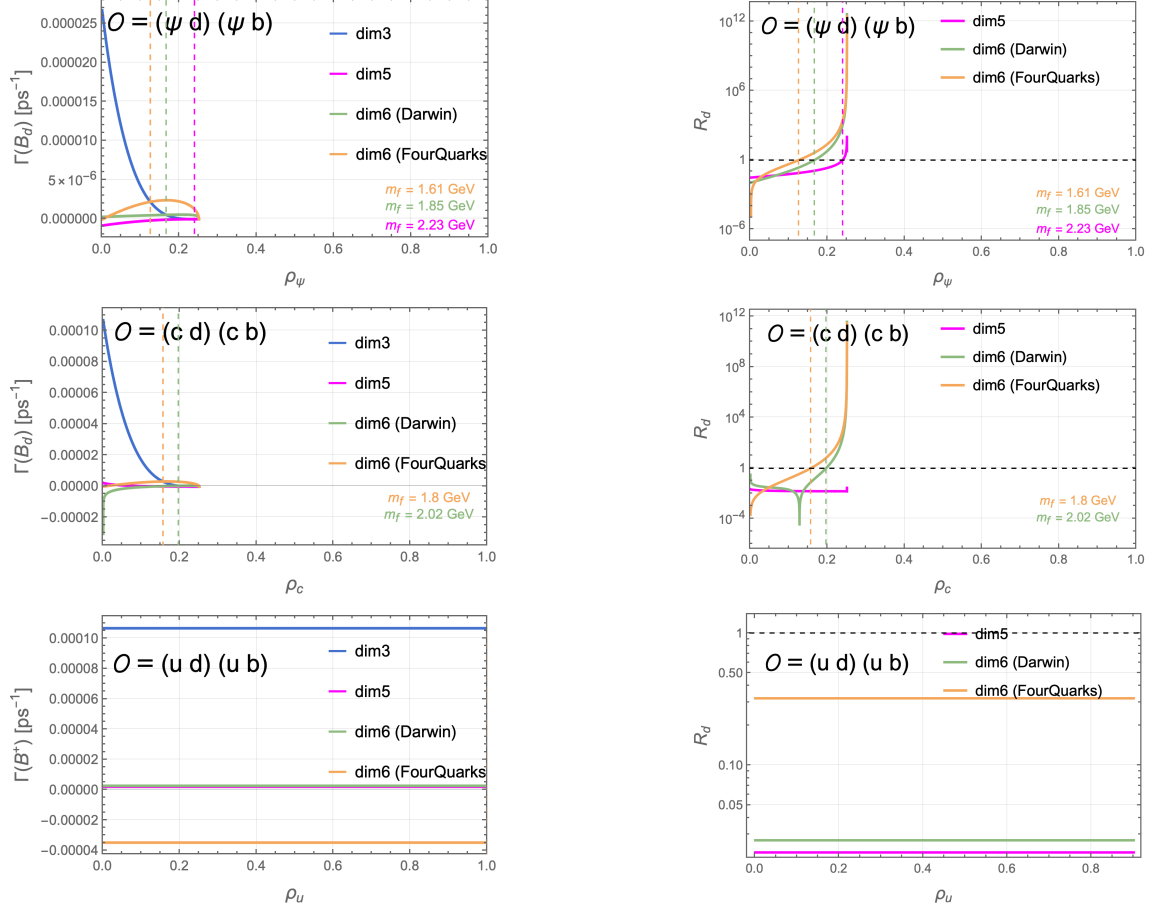


Figure D.3: (Part 1) Total decay width $\Gamma(B^+)$ induced by the Mesogenesis (BSM) operators as a function of $\rho_f = (m_f/m_b)^2$. The left and right columns show the individual HQE contributions and the corresponding ratios $R_d = |\Gamma_d/\Gamma_3|$, respectively. The vertical lines indicate the values of ρ_f (and the corresponding m_f) where a subleading contribution exceeds the leading dimension-three term Γ_3 . The curves terminate at the kinematic threshold where the loop particles can no longer be produced on shell, while the spikes in R_d arise from zeros of the corresponding subleading contribution.

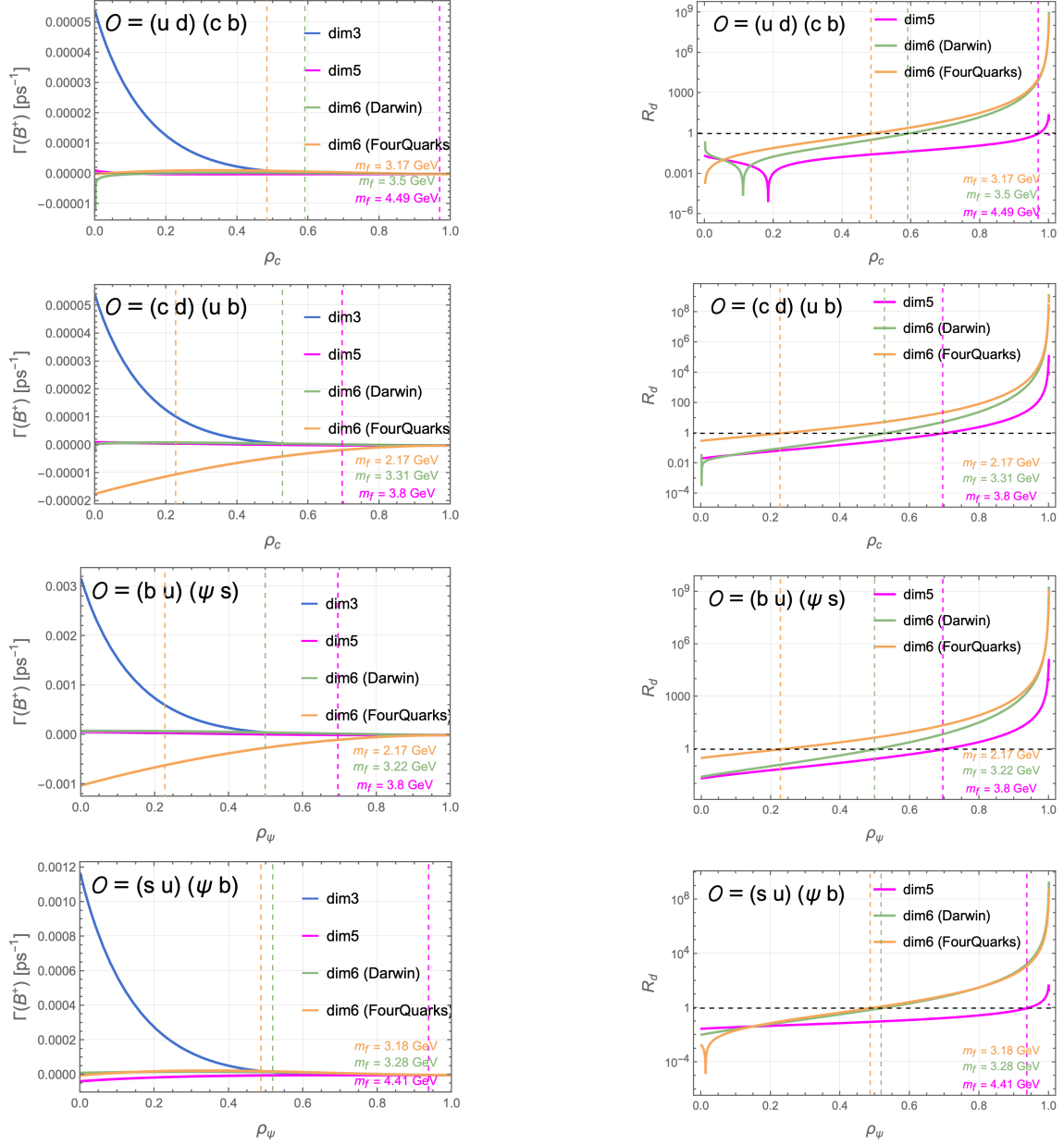


Figure D.4: (Part 2): Additional plots for different operators with the same colour coding as in the previous Figure.

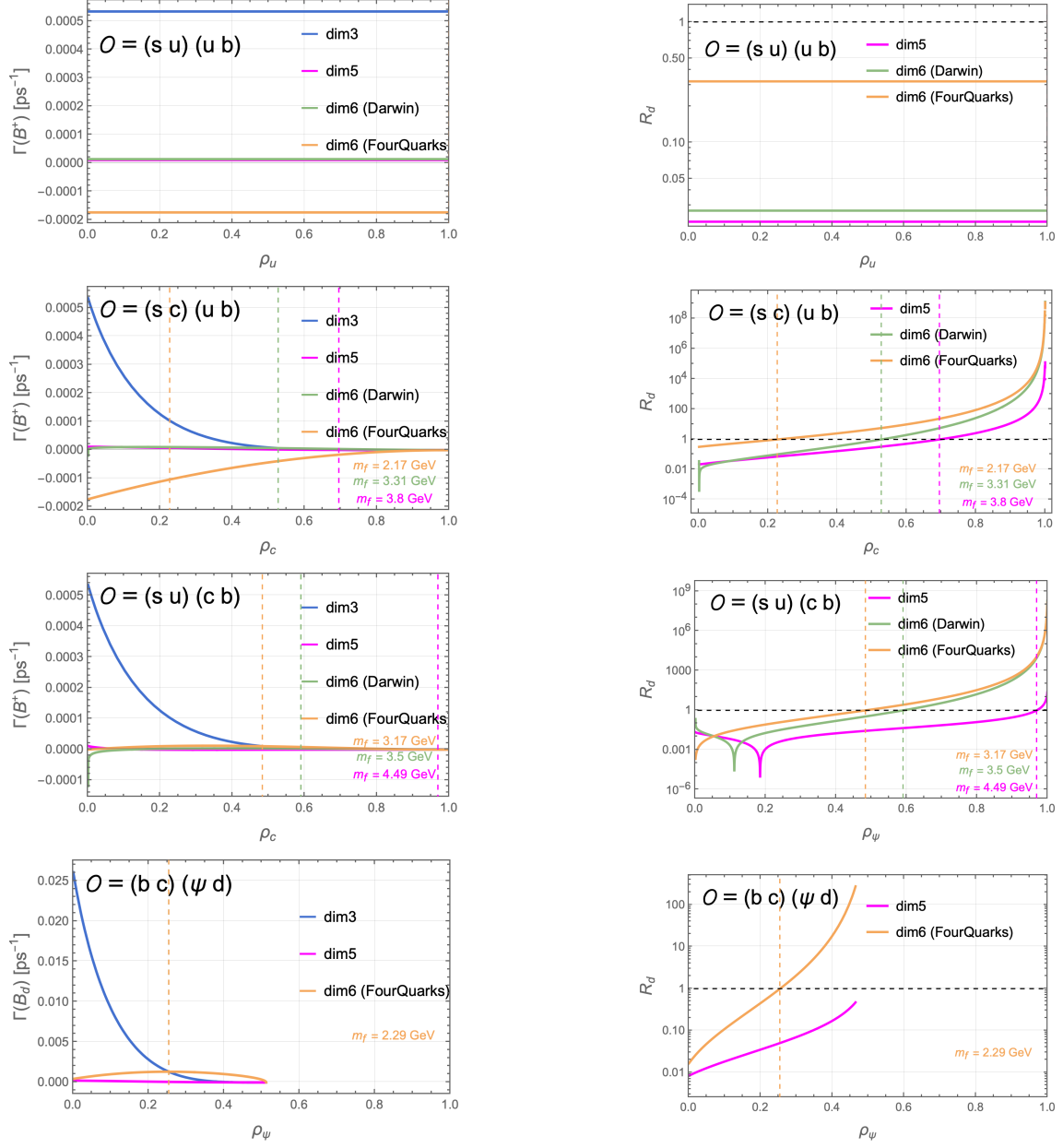


Figure D.5: (Part 3): Additional plots for different operators with the same colour coding as in the previous Figure.

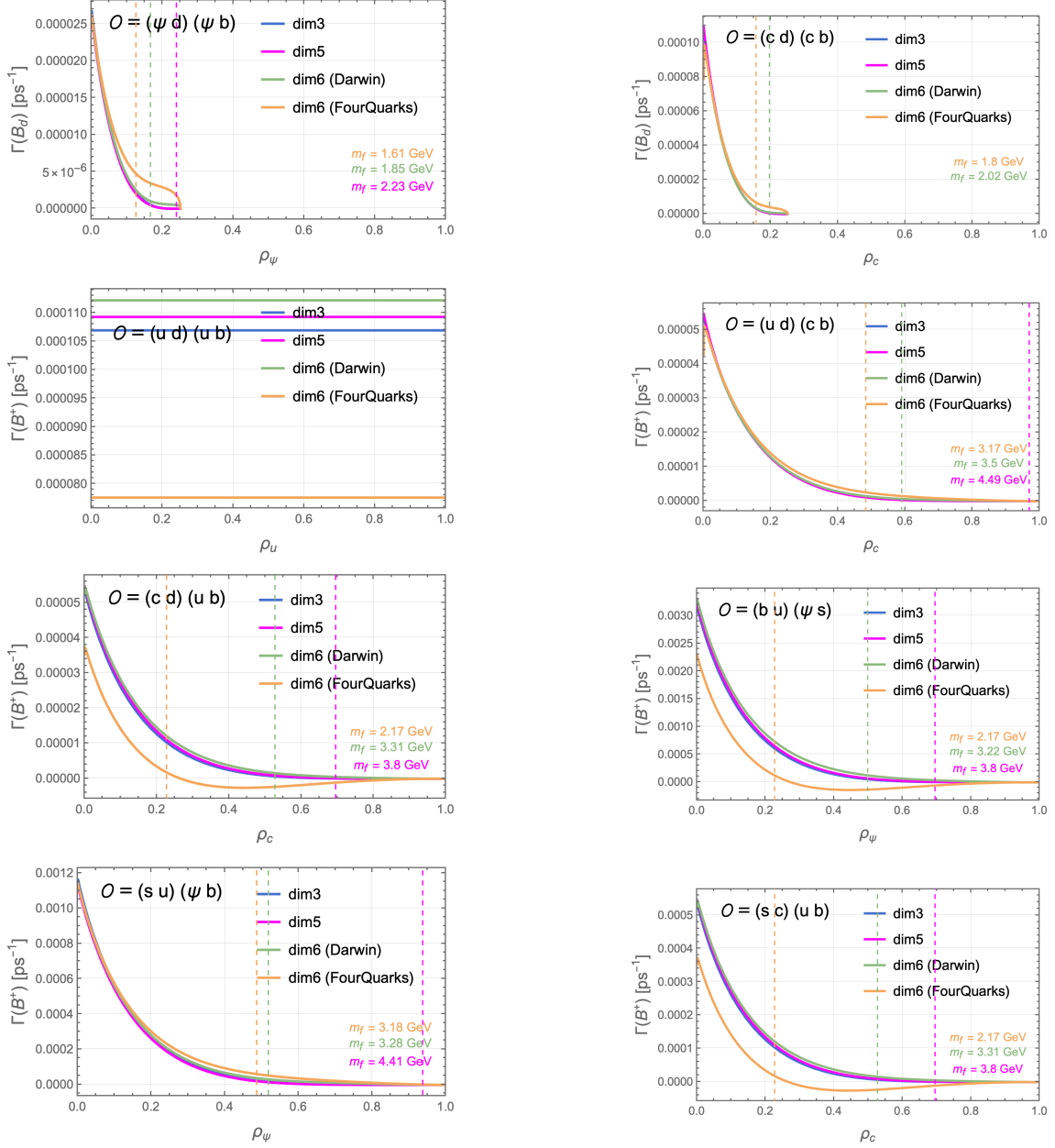


Figure D.6: (Part 1): Total decay width of $\Gamma(B^+)$ arising purely from the Mesogenesis (BSM) operators as a function of $\rho_f = (m_f/m_b)^2$. The figure displays the cumulative sum of all terms up to the indicated dimension. The vertical lines indicate the value of ρ_f , where the contribution exceeds the leading dimension-three Γ_3 term (the corresponding m_f values are also indicated).

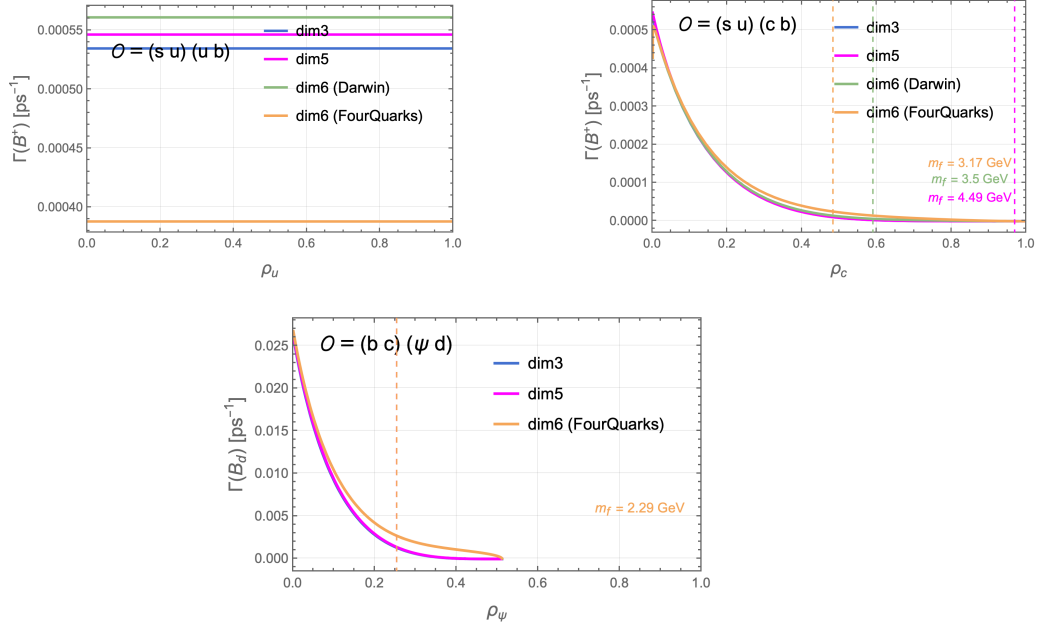


Figure D.7: (Part 2): Total decay width of $\Gamma(B^+)$ arising purely from the Mesogenesis (BSM) operators as a function of $\rho_f = (m_f/m_b)^2$, with the same colour coding as in the previous figure.

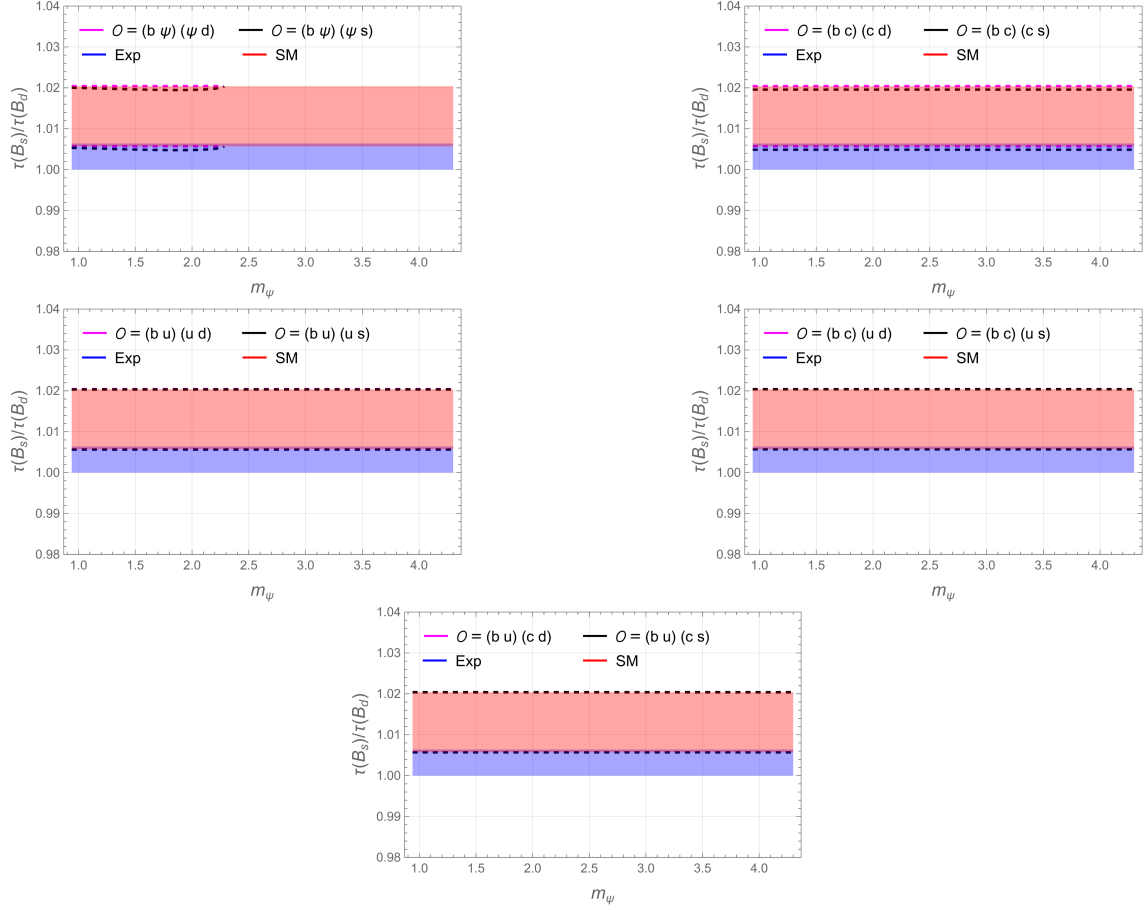


Figure D.8: Lifetime ratio $\tau(B_s)/\tau(B_d)$ as a function of the dark fermion mass m_ψ , including both SM and SM+NP predictions for representative operators. The NP contribution is found to be negligible compared to present theoretical uncertainties. (Operators involving two distinct massive final states are not shown, as the corresponding Darwin-term contribution has not yet been calculated.)

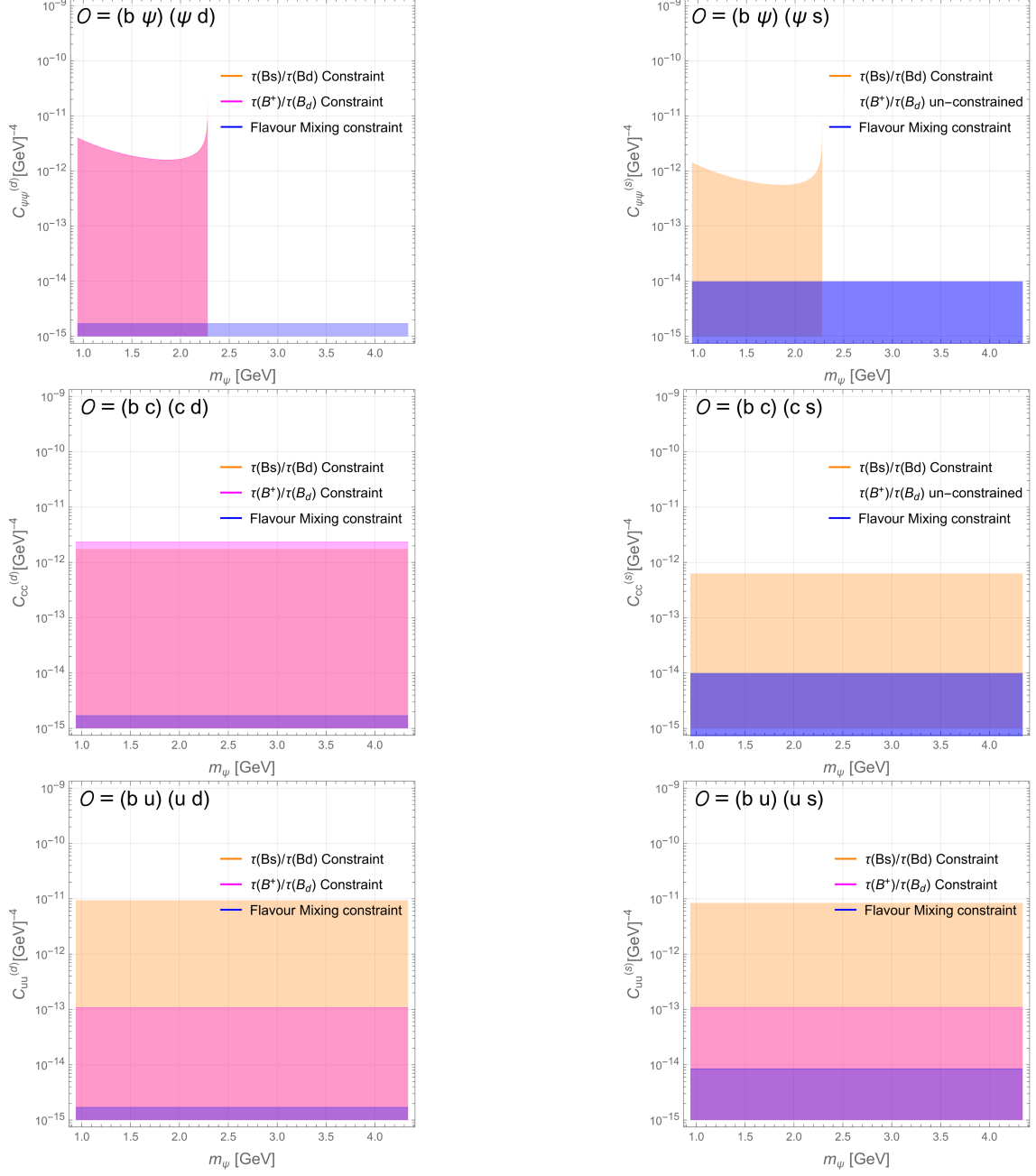


Figure D.9: (Part 1): Constraints on the Wilson coefficients as a function of the dark fermion mass m_ψ , derived from the lifetime ratios $\tau(B_s)/\tau(B_d)$ (orange) and $\tau(B^+)/\tau(B_d)$ (magenta), compared to collider constraints (blue).

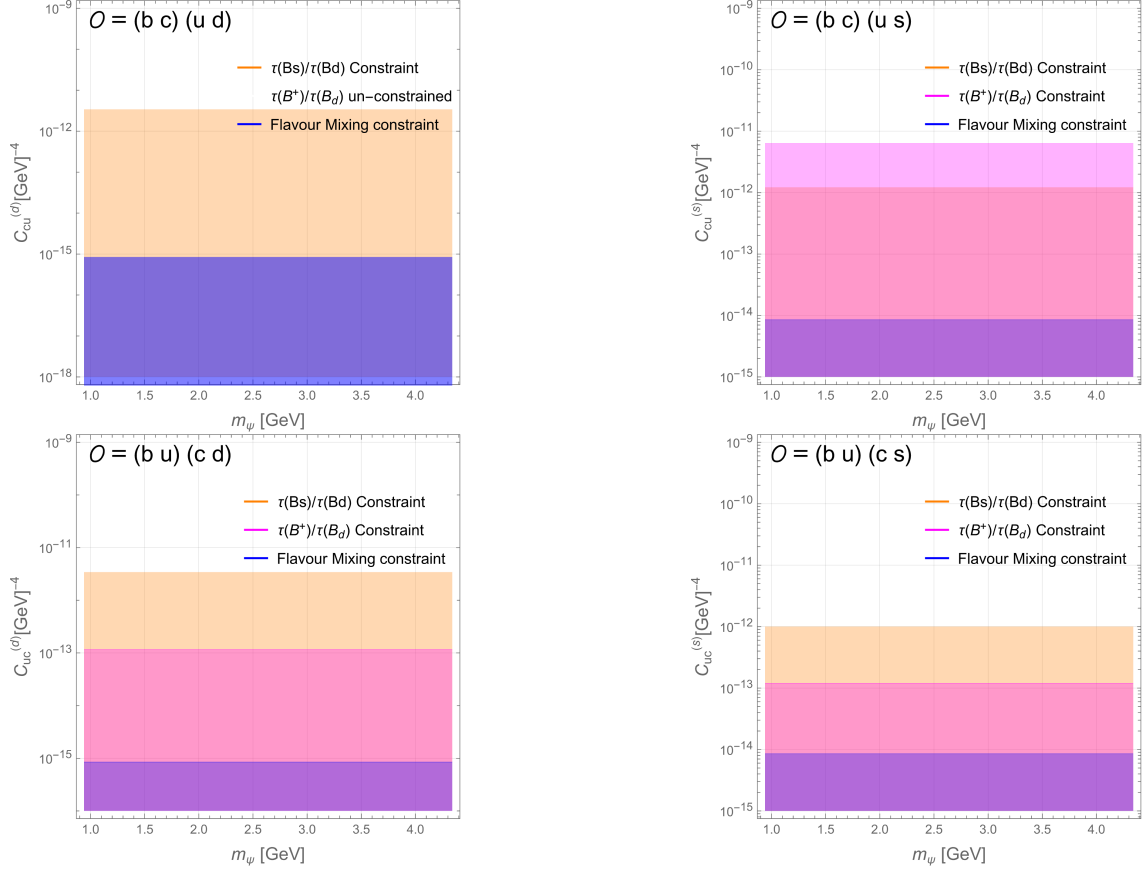


Figure D.10: (Part 2): Constraints on the Wilson coefficients as a function of the dark fermion mass m_ψ , with the same colour coding as in the previous figure. (Operators involving two distinct massive final states are not shown, as the corresponding Darwin-term contribution has not yet been calculated.)

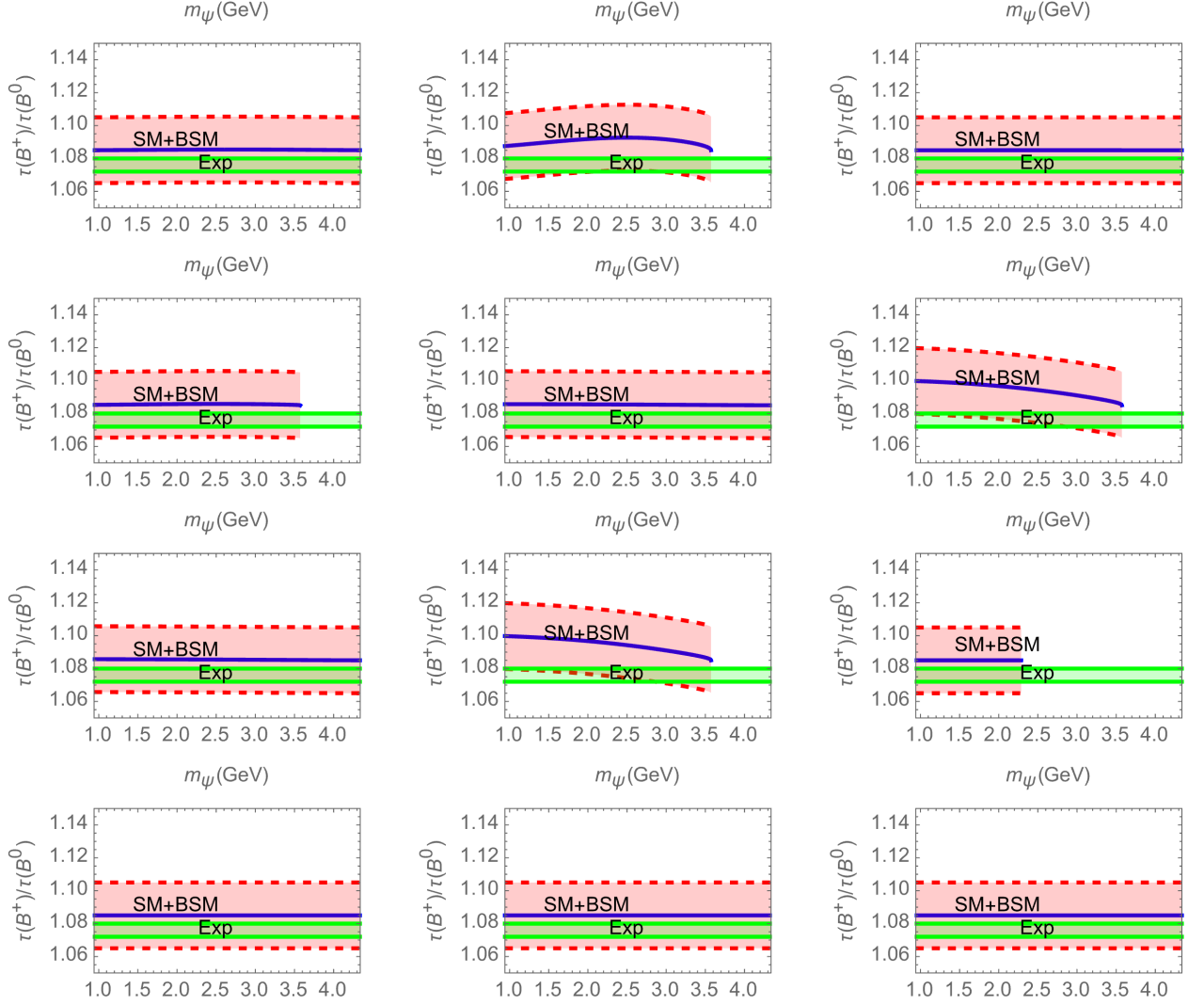


Figure D.11: The lifetime ratio $\tau(B^+)/\tau(B_d)$ for $\mathcal{H}_{-1/3}$ as a function of m_ψ , using the upper limits of one of the constants G_X^{q1q2} and setting the others to zero. The red region shows the SM error, whereas the experimental value is shown in green. The non-vanishing constants from up to down and left to right are $G_{III}^{\psi u}, G_{III}^{\psi c}, G_{III}^{u\psi}, G_{III}^{c\psi}, G_{II}^{\psi u}, G_{II}^{\psi c}, G_{II}^{u\psi}, G_{II}^{c\psi}, G_I^{\psi\psi}, G_I^{uu} = G_I^{cc}, G_I^{cu}, G_I^{uc}$

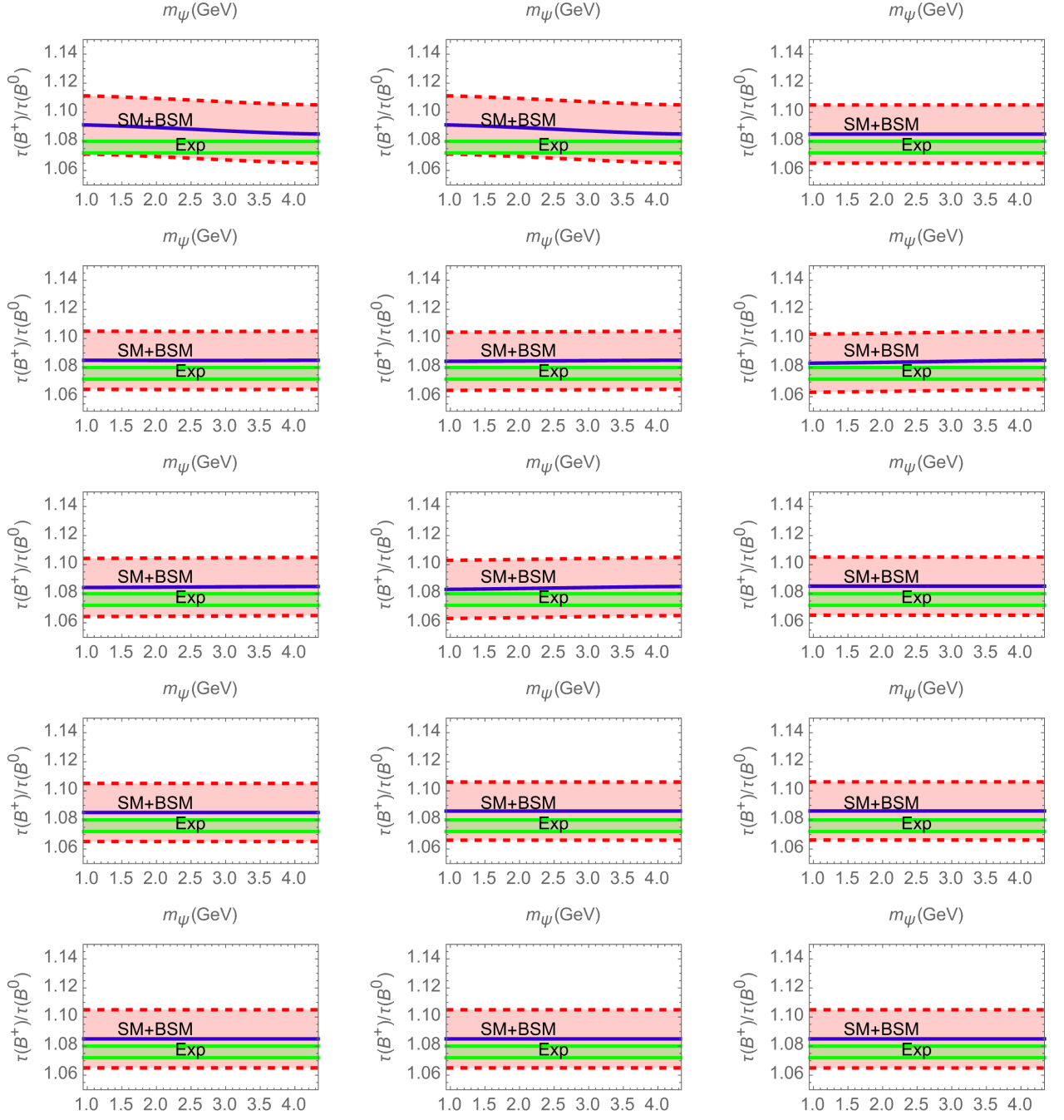


Figure D.12: The lifetime ratio $\tau(B^+)/\tau(B_d)$ for $\mathcal{H}_{-1/3}$ as a function of m_ψ , using the upper limits of one of the constants $G_X^{f_1 f_2}$ and setting the others to zero. The red region shows the SM error, whereas the experimental value is shown in green. The non-vanishing constants from up to down and left to right are $G_{IV}^{\psi d}, G_{IV}^{\psi s}, G_{III}^{\psi d}, G_{III}^{\psi s}, G_V^{\psi d}, G_V^{\psi s}, G_{VI}^{\psi d}, G_{VI}^{\psi s}, G_{IV}^{ud}, G_{IV}^{cd}, G_I^{sc}, G_{IV}^{su}, G_I^{du}, G_I^{dc}, G_I^{sc}, G_I^{su}$.

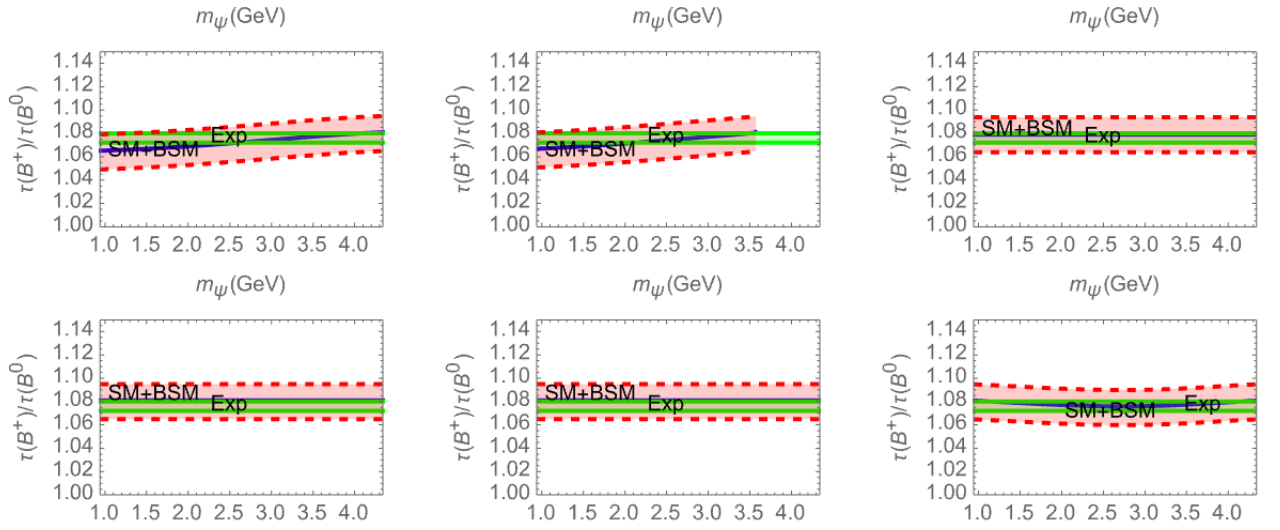


Figure D.13: The lifetime ratio $\tau(B^+)/\tau(B_d)$ for $\mathcal{H}_{2/3}$ as a function of m_ψ , using the upper limits of one of the constants $G_X^{f_1 f_2}$ and setting the others to zero. The red region shows the SM error, whereas the experimental value is shown in green. The non-vanishing constants from up to down and left to right are $G_{IV}^{u\psi}, G_{IV}^{c\psi}, G_{IV}^{ds}, G_I^{ds}, G_I^{ss} = G_{III}^{d\psi}, G_{III}^{s\psi}$,

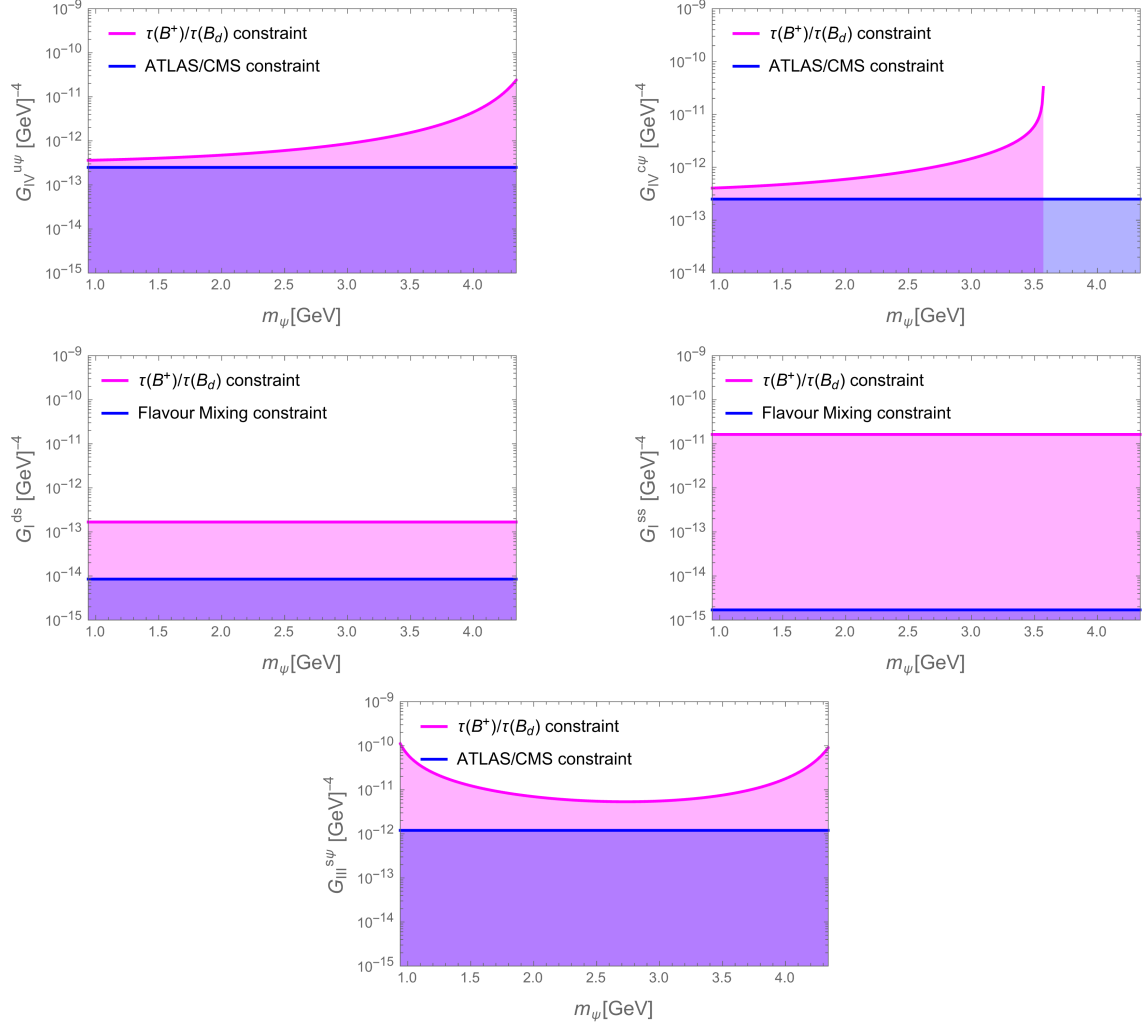


Figure D.14: $\mathcal{H}_{2/3}$: Constraints on the couplings $G_X^{f_1 f_2}$ derived from various sources. The magenta regions indicate the values allowed by the lifetime ratio $\tau(B^+)/\tau(B_d)$, while the blue region corresponds to the constraints obtained from collider signals or flavour mixing, as studied in [31].

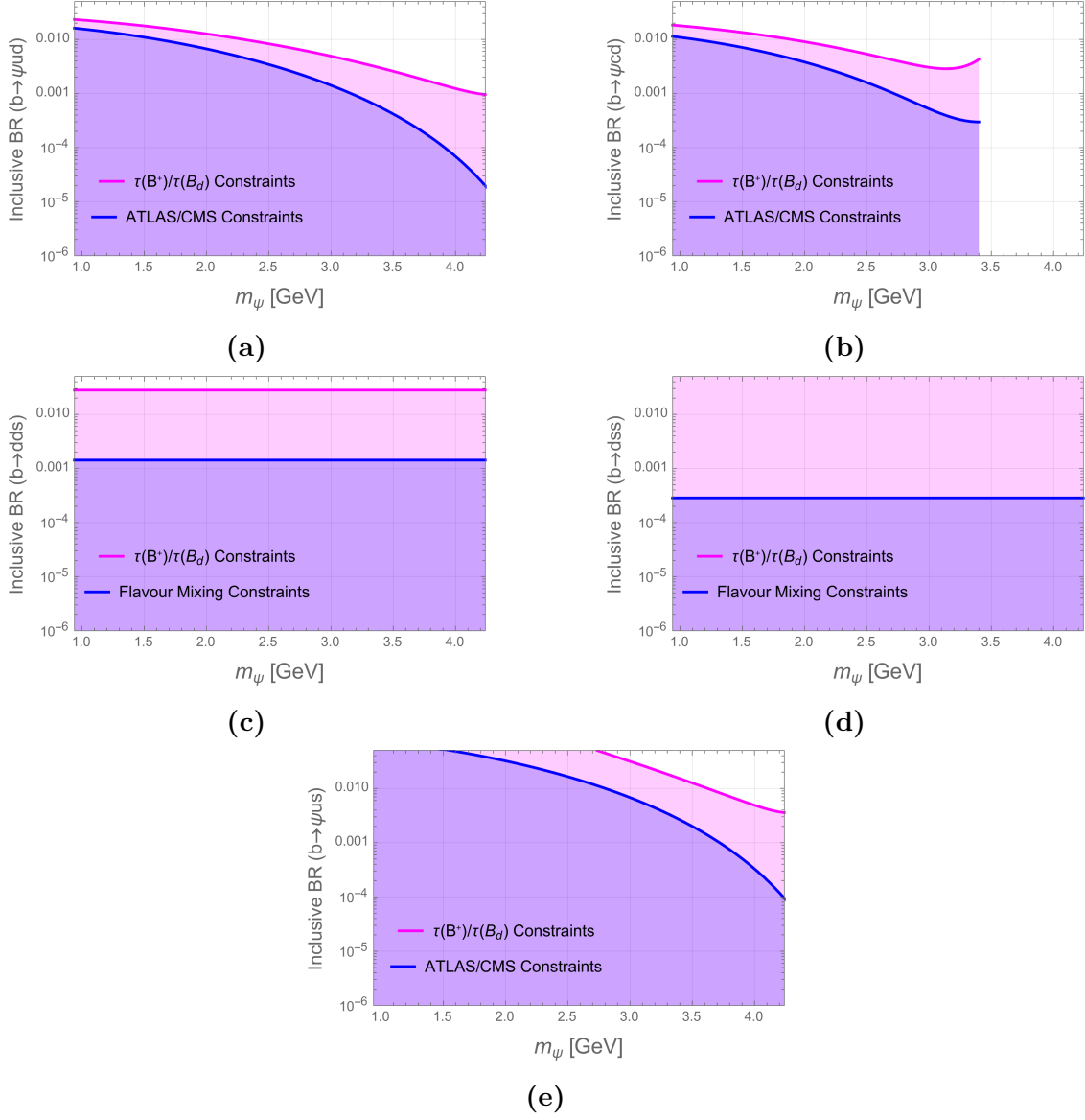


Figure D.15: $\mathcal{H}_{2/3}$: Allowed regions of the inclusive branching ratio of the different operators proportional to the couplings $G_X^{f_1 f_2}$. Values of the inclusive branching fraction above 5% are excluded by the bounds from the total decay rate of B mesons, therefore we do not show regions above 0.05 in the plots. The figure shows the difference between using the values allowed by the lifetime ratio $\tau(B^+)/\tau(B_d)$ (magenta) and those allowed by collider signals or flavour mixing (blue), as studied in [31], to compute the inclusive branching ratio. The plots correspond to the following couplings: (a) $G_{IV}^{u\psi} + G_{III}^{d\psi}$ (b) $G_{IV}^{c\psi}$ (c) $G_I^{ds} + G_{IV}^{ds}$ (d) G_I^{ss} (e) $G_{III}^{s\psi}$.

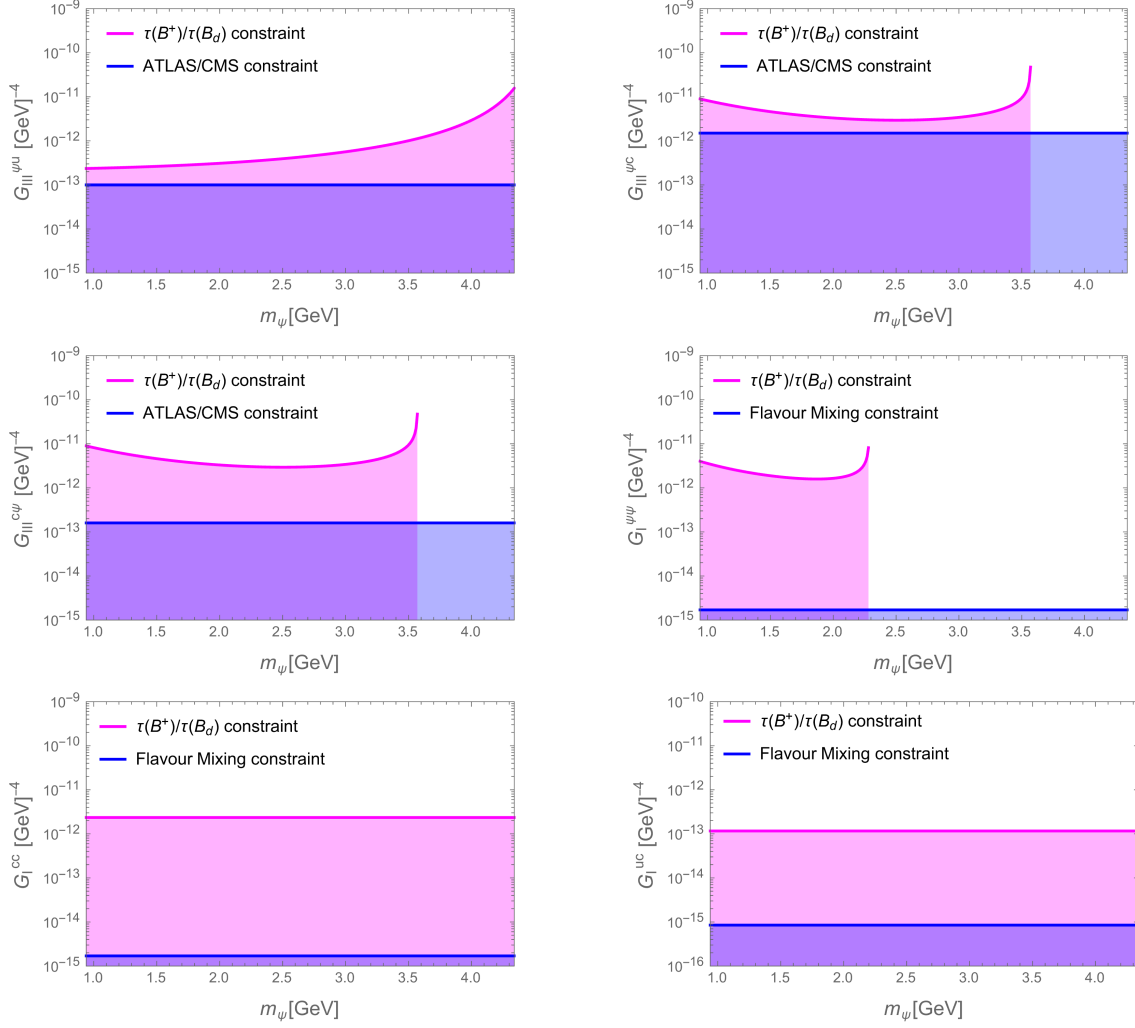


Figure D.16: $\mathcal{H}_{-1/3}$ (Part 1): Constraints on the couplings $G_X^{f_1 f_2}$ derived from various sources. The magenta regions indicate the values allowed by the lifetime ratio $\tau(B^+)/\tau(B_d)$, while the blue region corresponds to the constraints obtained from collider signals or flavour mixing, as studied in [31].

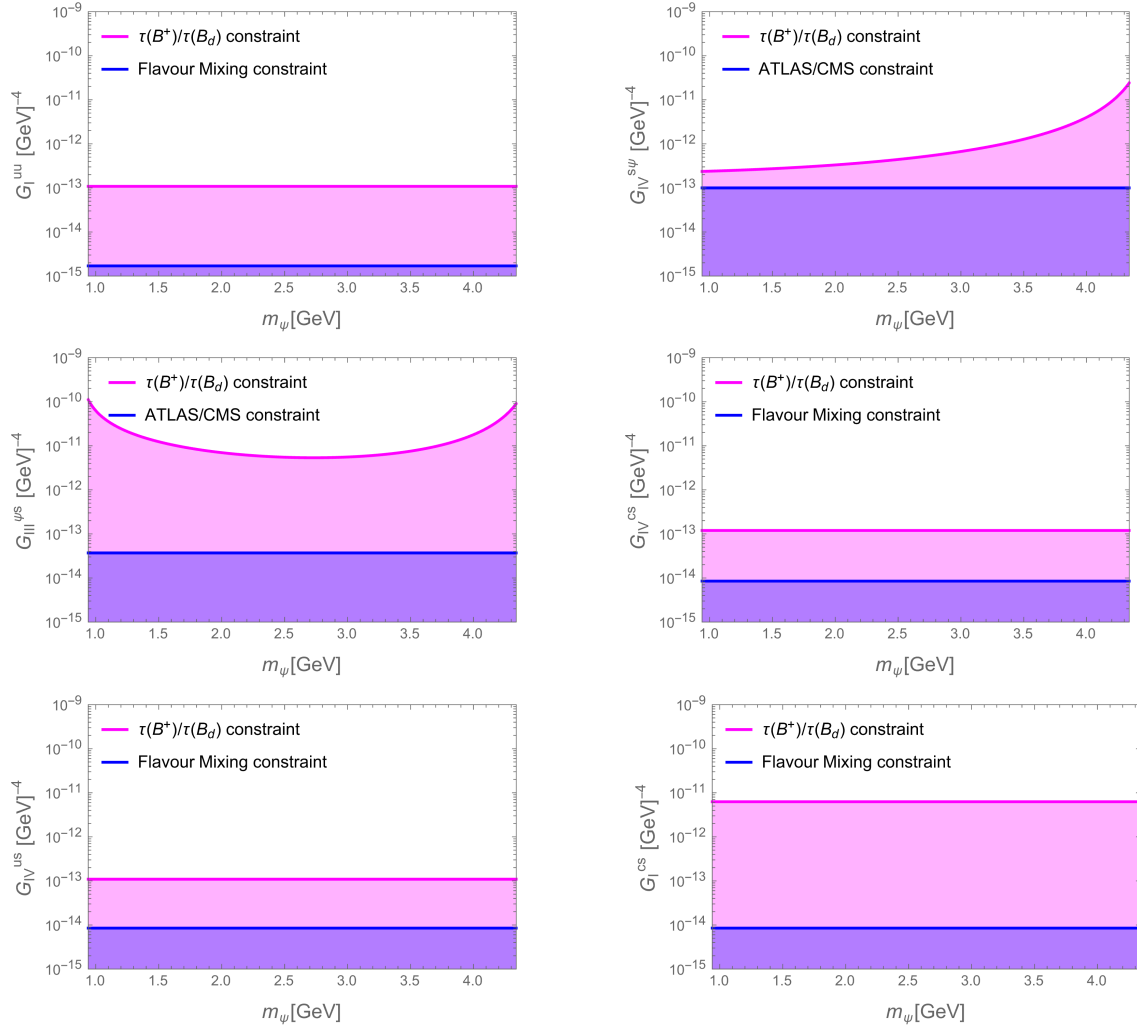


Figure D.17: $\mathcal{H}_{-1/3}$ (Part 2): Additional constraints on the couplings $G_X^{f_1 f_2}$ with the same colour coding as in the previous Figure.

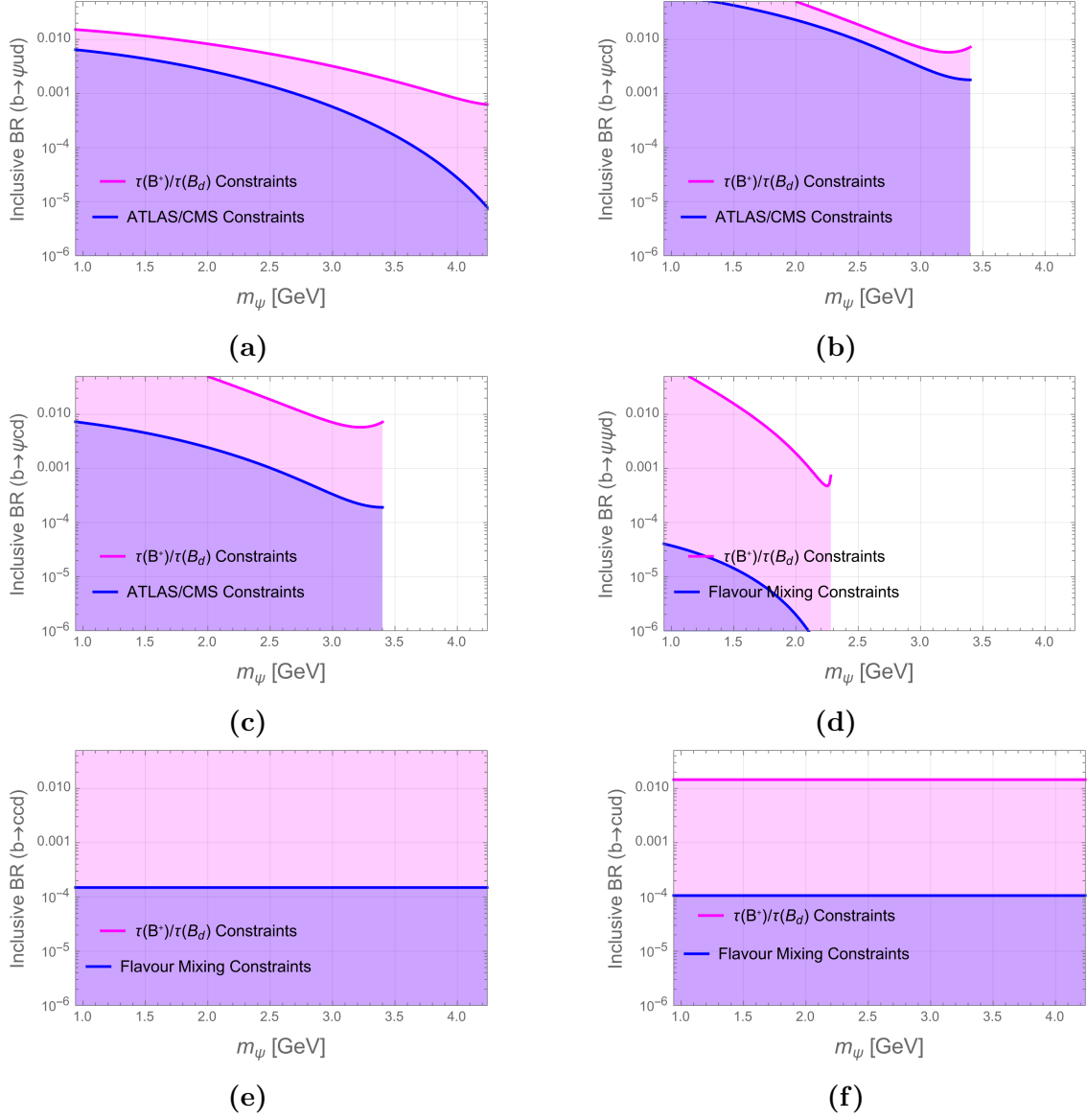


Figure D.18: $\mathcal{H}_{-1/3}$ (Part 1): The inclusive branching ratio of the different operators proportional to the couplings $G_X^{f_1 f_2}$. Values of the inclusive branching fraction above 5% are excluded by the bounds from the total decay rate of B mesons, therefore we do not show regions above 0.05 in the plots. The figure shows the difference between using the values allowed by the lifetime ratio $\tau(B^+)/\tau(B_d)$ and those allowed by collider signals or flavour mixing, as studied in [31], to compute the inclusive branching ratio. The plots correspond to the following couplings: (a) $G_{III}^{\psi u} + G_{IV}^{d\psi}$ (b) $G_{III}^{\psi c}$ (c) $G_{III}^{c\psi}$ (d) $G_I^{\psi\psi}$ (e) G_{IV}^{ud} (f) G_I^{cc} .

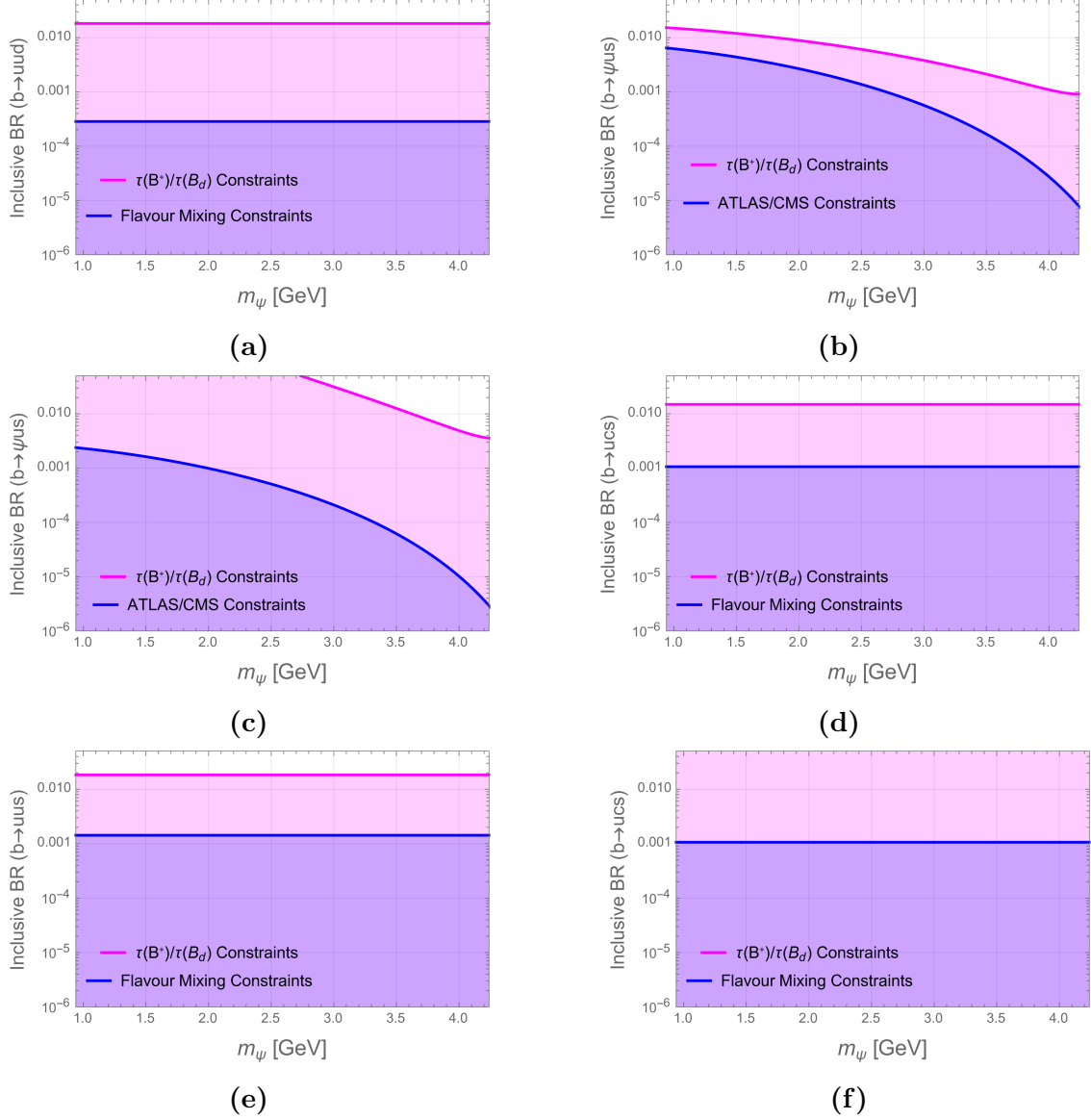


Figure D.19: $\mathcal{H}_{-1/3}$ (Part 2): Additional branching ratio plots with the same colour coding. Values of the inclusive branching fraction above 5% are excluded by the bounds from the total decay rate of B mesons, therefore we do not show regions above 0.05 in the plots. The plots correspond to the following couplings: (a) $G_I^{uc} + G_{IV}^{cd}$ (b) $G_{IV}^{s\psi}$ (c) $G_{III}^{\psi c}$ (d) $G_{IV}^{us} + G_I^{us}$ (e) G_{IV}^{cs} (f) G_I^{cs} .

Acknowledgments

One of the most important aspects of a doctoral journey is the supervisor. In this respect, I consider myself very fortunate. I first met my supervisor, Alexander Lenz, during a course he taught while I was a master's student in Siegen. It did not take long for me to decide that I wanted to pursue my PhD under his supervision, and I was delighted when he accepted me as his student.

Throughout my doctorate and master studies, he had an exceptional sense for choosing projects that challenged me without becoming overwhelming, allowing me to develop steadily as a researcher. At the same time, he gave me the freedom to grow independently. He trusted me to explore ideas on my own, to make mistakes, and to learn from them. Looking back, I realize that this trust was one of the greatest gifts a supervisor could have given me.

Beyond research, I was always grateful for his understanding of my personality. He never expected me to participate in every social event or group activity, but instead respected the way I preferred to work and interact with others. That level of trust and acceptance is something I believe is rare, and I deeply appreciate it.

Beyond his own scientific contributions, I have always admired Alex's commitment to strengthening the flavour physics community. He has devoted an enormous amount of time and energy to bringing physics to the public, inspiring young people to pursue the subject, and building the flavour physics group in Siegen into one of the leading groups in the field. Although many of these efforts naturally benefit the University of Siegen, their impact extends far beyond it. Investing in a strong and visible flavour physics community is, in the long run, an investment in the future of the field itself. Much of what exists today had to be built from the ground up, and I can only imagine how demanding that must have been. Despite these many responsibilities, he always found time to guide and collaborate with his students. Even when I pursued projects independently, he never hesitated to carefully read my manuscripts, provide detailed comments, and offer valuable advice. Through his example, he also taught me that being a good physicist extends beyond doing good research. Equally important is learning how to communicate one's work effectively, engage with the broader scientific community, and contribute to the growth of the field. For his trust, generosity with his time, and unwavering support, I will always be deeply grateful. On a personal level, I have always admired his attitude toward both success and setbacks. Whatever the circumstances, he would meet them with optimism, a smile, and simply keep moving forward. In that respect, I think we are quite similar—perhaps with the exception of the smile.

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PhD. For some reason I still do not fully understand, they always took me seriously. Every discussion with them left me with a much clearer idea of how to solve the problem I am having. In many respects, I learned the practical aspects of the Heavy Quark Expansion from them. Their guidance played a crucial role in my transition from being a student to becoming a researcher.

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Looking back, I realize that the broad education I received in Siegen was the result of learning from all of them. What makes the particle physics group in Siegen truly exceptional is not only its scientific excellence, but also its welcoming and collaborative atmosphere. Every office door was always open, and I never hesitated to knock on one for a discussion, advice, or simply a different perspective on a problem. Although I was fortunate to have Alex as my supervisor, I always felt supported by the entire group. In that sense, I benefited from the guidance of far more people than one could normally expect during a PhD. As Damir Bećirević once told me, “You are in a good group, of good people, in good hands.” I could not agree more. I would also like to thank Gudrun Hiller and Miguel Escudero for the many fruitful discussions and the continuous exchange of ideas. Together with Maria Laura, they made my visit to CERN possible and created an exceptionally welcoming and inspiring environment. Thanks to the three of them, my time at CERN became one of the most memorable and rewarding experiences of my PhD.

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Throughout my life, I have been inspired by the achievements of many remarkable scientists. Yet, with time, I came to realize that the achievements which have shaped my own life the most are not those found in textbooks or scientific discoveries, but the countless sacrifices made by my parents. Through their own example, they taught me how much one is willing to do for the happiness and future of their children. They also taught me that no matter what one achieves in life, there is always room to grow, to learn, and to become a better person.

Of all the countless gifts they gave me, the greatest was, in my opinion, freedom. They gave me the freedom to discover who I wanted to become, to make my own choices, and to follow my own path. More importantly, they created an environment in which I could exercise that freedom with confidence, knowing that I would always have their unconditional love and support. When I decided to pursue physics, they supported me wholeheartedly, even though the details of my work are far removed from their own experiences. They never asked me to become what they wanted me to be; instead, they encouraged me to become who I wanted to be. That is the greatest gift I have ever received, and one I will cherish for the rest of my life.

Bibliography

- [1] S. L. Glashow, *Partial Symmetries of Weak Interactions*, *Nucl. Phys.* **22** (1961) 579–588.
- [2] S. Weinberg, *A Model of Leptons*, *Phys. Rev. Lett.* **19** (1967) 1264–1266.
- [3] A. Salam, *Weak and Electromagnetic Interactions*, in *Elementary particle physics: relativistic groups and analyticity* (N. Svartholm, ed.), vol. 680519, pp. 367–377, 1968.
- [4] **ATLAS** Collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1–29, [[arXiv:1207.7214](#)].
- [5] **CMS** Collaboration, S. Chatrchyan et al., *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30–61, [[arXiv:1207.7235](#)].
- [6] F. Englert and R. Brout, *Broken Symmetry and the Mass of Gauge Vector Mesons*, *Phys. Rev. Lett.* **13** (1964) 321–323.
- [7] P. W. Higgs, *Broken Symmetries and the Masses of Gauge Bosons*, *Phys. Rev. Lett.* **13** (1964) 508–509.
- [8] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *Global Conservation Laws and Massless Particles*, *Phys. Rev. Lett.* **13** (1964) 585–587.
- [9] A. D. Sakharov, *Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe*, *Pisma Zh. Eksp. Teor. Fiz.* **5** (1967) 32–35. [*Usp. Fiz. Nauk*161,61(1991)].
- [10] F. Zwicky, *Die Rotverschiebung von extragalaktischen Nebeln*, *Helv. Phys. Acta* **6** (1933) 110–127.
- [11] V. C. Rubin and W. K. Ford, Jr., *Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions*, *Astrophys. J.* **159** (1970) 379–403.
- [12] **Supernova Search Team** Collaboration, A. G. Riess et al., *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, *Astron. J.* **116** (1998) 1009–1038, [[astro-ph/9805201](#)].

- [13] **Supernova Cosmology Project** Collaboration, S. Perlmutter et al., *Measurements of Omega and Lambda from 42 high redshift supernovae*, *Astrophys. J.* **517** (1999) 565–586, [[astro-ph/9812133](#)].
- [14] W. Altmannshofer and P. Stangl, *New physics in rare B decays after Moriond 2021*, *Eur. Phys. J. C* **81** (2021), no. 10 952, [[arXiv:2103.13370](#)].
- [15] S. Bifani, S. Descotes-Genon, A. Romero Vidal, and M.-H. Schune, *Review of Lepton Universality tests in B decays*, *J. Phys. G* **46** (2019), no. 2 023001, [[arXiv:1809.06229](#)].
- [16] P. Langacker, *The Physics of Heavy Z-prime Gauge Bosons*, *Rev. Mod. Phys.* **81** (2009) 1199–1228, [[arXiv:0801.1345](#)].
- [17] G. Isidori, Y. Nir, and G. Perez, *Flavor Physics Constraints for Physics Beyond the Standard Model*, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 355–380, [[arXiv:1002.0900](#)].
- [18] N. Cabibbo, *Unitary Symmetry and Leptonic Decays*, *Phys. Rev. Lett.* **10** (1963) 531–533.
- [19] M. Kobayashi and T. Maskawa, *CP Violation in the Renormalizable Theory of Weak Interaction*, *Prog. Theor. Phys.* **49** (1973) 652–657.
- [20] C. D. Froggatt and H. B. Nielsen, *Hierarchy of Quark Masses, Cabibbo Angles and CP Violation*, *Nucl. Phys. B* **147** (1979) 277–298.
- [21] S. L. Glashow, J. Iliopoulos, and L. Maiani, *Weak Interactions with Lepton-Hadron Symmetry*, *Phys. Rev.* **D2** (1970) 1285–1292.
- [22] G. D’Ambrosio, G. F. Giudice, G. Isidori, and A. Strumia, *Minimal flavor violation: An Effective field theory approach*, *Nucl. Phys. B* **645** (2002) 155–187, [[hep-ph/0207036](#)].
- [23] M. A. Shifman and M. B. Voloshin, *Preasymptotic Effects in Inclusive Weak Decays of Charmed Particles*, *Sov. J. Nucl. Phys.* **41** (1985) 120. [*Yad. Fiz.* 41,187(1985)].
- [24] J. Chay, H. Georgi, and B. Grinstein, *Lepton energy distributions in heavy meson decays from QCD*, *Phys. Lett. B* **247** (1990) 399–405.
- [25] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev, and A. I. Vainshtein, *The Blessings of the $1/m_Q$ expansions in inclusive semileptonic B decays and their challenges*, *Int. J. Mod. Phys. A* **9** (1994) 2467–2504, [[hep-ph/9312359](#)].
- [26] A. V. Manohar and M. B. Wise, *Inclusive semileptonic B and polarized Lambda(b) decays from QCD*, *Phys. Rev. D* **49** (1994) 1310–1329, [[hep-ph/9308246](#)].
- [27] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nonleptonic Decays of K Mesons and Hyperons*, *Sov. Phys. JETP* **45** (1977) 670. [*Zh. Eksp. Teor. Fiz.* 72,1275(1977)].

- [28] A. J. Buras and M. Munz, *Effective Hamiltonian for $B \rightarrow X_s e^+ e^-$ beyond leading logarithms in the NDR and HV schemes*, *Phys. Rev. D* **52** (1995) 186–195, [[hep-ph/9501281](#)].
- [29] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, *Weak decays beyond leading logarithms*, *Rev. Mod. Phys.* **68** (1996) 1125–1144, [[hep-ph/9512380](#)].
- [30] G. Elor, M. Escudero, and A. Nelson, *Baryogenesis and Dark Matter from B Mesons*, *Phys. Rev. D* **99** (2019), no. 3 035031, [[arXiv:1810.00880](#)].
- [31] G. Alonso-Álvarez, G. Elor, and M. Escudero, *Collider signals of baryogenesis and dark matter from B mesons: A roadmap to discovery*, *Phys. Rev. D* **104** (2021), no. 3 035028, [[arXiv:2101.02706](#)].
- [32] F. Elahi, G. Elor, and R. McGehee, *Charged B mesogenesis*, *Phys. Rev. D* **105** (2022), no. 5 055024, [[arXiv:2109.09751](#)].
- [33] A. E. Nelson and H. Xiao, *Baryogenesis from B Meson Oscillations*, *Phys. Rev. D* **100** (2019), no. 7 075002, [[arXiv:1901.08141](#)].
- [34] T. Mannel, D. Moreno, and A. A. Pivovarov, *QCD corrections for subleading powers in $1/\text{mb}$ for the nonleptonic $b \rightarrow c\bar{c}s$ transition*, *Phys. Rev. D* **111** (2025), no. 9 094035, [[arXiv:2503.18775](#)].
- [35] T. Mannel, D. Moreno, and A. A. Pivovarov, *QCD corrections at subleading power for inclusive nonleptonic $b \rightarrow c\bar{u}d$ decays*, *Phys. Rev. D* **110** (2024), no. 9 094011, [[arXiv:2408.06767](#)].
- [36] M. Egner, M. Fael, K. Schönwald, and M. Steinhauser, *Nonleptonic B-meson decays to next-to-next-to-leading order*, *JHEP* **10** (2024) 144, [[arXiv:2406.19456](#)]. [Erratum: JHEP 02, 147 (2025)].
- [37] F. Moretti, U. Nierste, P. Reeck, and M. Steinhauser, *Next-to-next-to-leading QCD corrections to the $\mathbf{B}^+ - \mathbf{B}_d^0$, $\mathbf{D}^+ - \mathbf{D}^0$, and $\mathbf{D}_s^+ - \mathbf{D}^0$ lifetime ratios*, [arXiv:2604.24841](#).
- [38] M. Black, M. Lang, A. Lenz, and Z. Wüthrich, *HQET sum rules for matrix elements of dimension-six four-quark operators for meson lifetimes within and beyond the Standard Model*, *JHEP* **04** (2025) 081, [[arXiv:2412.13270](#)].
- [39] M. Black, R. V. Harlander, J. T. Kohnen, F. Lange, A. Rago, A. Shindler, and O. Witzel, *Bag Parameters for Heavy Meson Lifetimes*, [arXiv:2603.28516](#).
- [40] M. Black, R. V. Harlander, J. T. Kohnen, F. Lange, A. Rago, A. Shindler, and O. Witzel, *Heavy-Meson Bag Parameters using Gradient Flow*, [arXiv:2603.28517](#).
- [41] A. Lenz, *Lifetimes and heavy quark expansion*, *Int. J. Mod. Phys. A* **30** (2015), no. 10 1543005, [[arXiv:1405.3601](#)].

- [42] J. Albrecht, F. Bernlochner, A. Lenz, and A. Rusov, *Lifetimes of b -hadrons and mixing of neutral B -mesons: theoretical and experimental status*, *Eur. Phys. J. ST* **233** (2024), no. 2 359–390, [[arXiv:2402.04224](#)].
- [43] A. Khodjamirian, B. Melić, and Y.-M. Wang, *A guide to the QCD light-cone sum rules for b -quark decays*, *Eur. Phys. J. ST* **233** (2024), no. 2 271–298, [[arXiv:2311.08700](#)].
- [44] V. M. Braun and A. Lenz, *On the $SU(3)$ symmetry-breaking corrections to meson distribution amplitudes*, *Phys. Rev. D* **70** (2004) 074020, [[hep-ph/0407282](#)].
- [45] P. Ball and R. Zwicky, *$B_{d,s} \rightarrow \rho, \omega, K^*, \phi$ decay form-factors from light-cone sum rules revisited*, *Phys. Rev. D* **71** (2005) 014029, [[hep-ph/0412079](#)].
- [46] S. Weinberg, *Phenomenological Lagrangians*, *Physica A* **96** (1979), no. 1-2 327–340.
- [47] K. G. Wilson, *Nonlagrangian models of current algebra*, *Phys. Rev.* **179** (1969) 1499–1512.
- [48] A. V. Manohar, *Introduction to Effective Field Theories*, *Les Houches Summer School: EFT in Particle Physics and Cosmology* (2018) [[arXiv:1804.05863](#)].
- [49] A. J. Buras, *Weak Hamiltonian, CP violation and rare decays*, in *Les Houches Summer School in Theoretical Physics, Session 68: Probing the Standard Model of Particle Interactions*, pp. 281–539, 6, 1998. [hep-ph/9806471](#).
- [50] K. G. Chetyrkin, M. Misiak, and M. Münz, *Beta functions and anomalous dimensions up to three loops*, *Nucl. Phys. B* **518** (1998) 473–494, [[hep-ph/9711266](#)].
- [51] M. Gorbahn and U. Haisch, *Effective Hamiltonian for non-leptonic $|\Delta F| = 1$ decays at NNLO in QCD*, *Nucl. Phys. B* **713** (2005) 291–332, [[hep-ph/0411071](#)].
- [52] K. G. Wilson and J. B. Kogut, *The Renormalization group and the epsilon expansion*, *Phys. Rept.* **12** (1974) 75–199.
- [53] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Wilson’s Operator Expansion: Can It Fail?*, *Nucl. Phys. B* **249** (1985) 445–471.
- [54] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *QCD and Resonance Physics: Applications*, *Nucl. Phys. B* **147** (1979) 385–447.
- [55] V. Fock, *Proper time in classical and quantum mechanics*, *Phys. Z. Sowjetunion* **12** (1937) 404–425.
- [56] J. S. Schwinger, *On gauge invariance and vacuum polarization*, *Phys. Rev.* **82** (1951) 664–679.

- [57] M. S. Dubovikov and A. V. Smilga, *Analytical Properties of the Quark Polarization Operator in an External Selfdual Field*, *Nucl. Phys. B* **185** (1981) 109–132.
- [58] E. V. Shuryak and A. I. Vainshtein, *Theory of Power Corrections to Deep Inelastic Scattering in Quantum Chromodynamics. 1. Q^2 Effects*, *Nucl. Phys. B* **199** (1982) 451–481.
- [59] A. V. Smilga, *Calculation of the degree corrections in fixed point gauge. (in Russian)*, *Sov. J. Nucl. Phys.* **35** (1982) 271–277.
- [60] M. A. Shifman, *Lectures on heavy quarks in quantum chromodynamics*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95): QCD and Beyond*, 10, 1995. [hep-ph/9510377](https://arxiv.org/abs/hep-ph/9510377).
- [61] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Calculations in External Fields in Quantum Chromodynamics. Technical Review*, *Fortsch. Phys.* **32** (1984) 585.
- [62] M. L. Piscopo, *Higher order corrections to the lifetime of heavy hadrons*. PhD thesis, Siegen U., 2021. [arXiv:2112.03137](https://arxiv.org/abs/2112.03137).
- [63] A. Lenz, M. L. Piscopo, and A. V. Rusov, *Contribution of the Darwin operator to non-leptonic decays of heavy quarks*, *JHEP* **12** (2020) 199, [[arXiv:2004.09527](https://arxiv.org/abs/2004.09527)].
- [64] A. V. Rusov, *Higher-twist effects in light-cone sum rule for the $B \rightarrow \pi$ form factor*, *Eur. Phys. J. C* **77** (2017), no. 7 442, [[arXiv:1705.01929](https://arxiv.org/abs/1705.01929)].
- [65] W. Chen, Z.-X. Cai, and S.-L. Zhu, *Masses of the tensor mesons with $J^P = 2^-$* , *Nucl. Phys. B* **887** (2014) 201–215, [[arXiv:1107.4949](https://arxiv.org/abs/1107.4949)].
- [66] A. Khodjamirian, *Hadron Form Factors*. CRC Press, 4, 2020.
- [67] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *QCD and Resonance Physics: Applications*, *Nucl. Phys. B* **147** (1979) 448–518.
- [68] I. I. Balitsky, V. M. Braun, and A. V. Kolesnichenko, *Radiative Decay $\Sigma^+ \rightarrow p\gamma$ in Quantum Chromodynamics*, *Nucl. Phys. B* **312** (1989) 509–550.
- [69] I. I. Balitsky and V. M. Braun, *The Role of QCD Vacuum Condensates in the Photon Photon and Photon Meson Reactions*, *Nucl. Phys. B* **311** (1989) 541–584.
- [70] K. G. Wilson, *Confinement of Quarks*, *Phys. Rev. D* **10** (1974) 2445–2459.
- [71] **Flavour Lattice Averaging Group (FLAG) Collaboration**, Y. Aoki et al., *FLAG Review 2024*, [arXiv:2411.04268](https://arxiv.org/abs/2411.04268).
- [72] K. G. Chetyrkin and F. V. Tkachov, *Integration by parts: The algorithm to calculate β -functions in 4 loops*, *Nucl. Phys. B* **192** (1981) 159–204.

- [73] S. Laporta, *High-precision calculation of multiloop feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087–5159.
- [74] E. Remiddi and L. Tancredi, *Differential equations and dispersion relations for Feynman amplitudes. The two-loop massive sunrise and the kite integral*, *Nucl. Phys.* **B907** (2016) 400–444, [[arXiv:1602.01481](#)].
- [75] A. A. Pivovarov, N. N. Tavkhelidze, and V. F. Tokarev, *The Schwinger model and sum rules*, *Phys. Lett.* **132B** (1983) 402–406.
- [76] S. Groote and A. A. Pivovarov, *Threshold expansion of Feynman diagrams within a configuration space technique*, *Nucl. Phys.* **B580** (2000) 459–484, [[hep-ph/0003115](#)].
- [77] J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, *Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism*, *JHEP* **05** (2018) 093, [[arXiv:1712.07089](#)].
- [78] J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, *Elliptic polylogarithms and iterated integrals on elliptic curves II: an application to the sunrise integral*, *Phys. Rev.* **D97** (2018), no. 11 116009, [[arXiv:1712.07095](#)].
- [79] T. Mannel, I. S. Milutin, and K. K. Vos, *Inclusive semileptonic $b \rightarrow c \ell \bar{\nu}$ decays to order $1/m_b^5$* , *JHEP* **02** (2024) 226, [[arXiv:2311.12002](#)]. [Erratum: *JHEP* **02**, 167 (2025)].
- [80] F. Gabbiani, A. I. Onishchenko, and A. A. Petrov, *Spectator effects and lifetimes of heavy hadrons*, *Phys. Rev.* **D70** (2004) 094031, [[hep-ph/0407004](#)].
- [81] A. Lenz, J. Müller, M. L. Piscopo, and A. V. Rusov, *Taming new physics in $b \rightarrow c \bar{u} d, s$ with $\tau(B^+)/\tau(B_d)$ and a_{sl}^d* , *JHEP* **09** (2023) 028, [[arXiv:2211.02724](#)].
- [82] M. Ciuchini, E. Franco, V. Lubicz, G. Martinelli, I. Scimemi, and L. Silvestrini, *Next-to-leading order QCD corrections to $\Delta F = 2$ effective Hamiltonians*, *Nucl. Phys. B* **523** (1998) 501–525, [[hep-ph/9711402](#)].
- [83] A. J. Buras, M. Misiak, and J. Urban, *Two loop QCD anomalous dimensions of flavor changing four quark operators within and beyond the standard model*, *Nucl. Phys. B* **586** (2000) 397–426, [[hep-ph/0005183](#)]. [Erratum: *Nucl.Phys.B* **1002**, 116529 (2024)].
- [84] S. Jäger, M. Kirk, A. Lenz, and K. Leslie, *Charming new physics in rare B -decays and mixing?*, *Phys. Rev. D* **97** (2018), no. 1 015021, [[arXiv:1701.09183](#)].
- [85] S. Jäger, M. Kirk, A. Lenz, and K. Leslie, *Charming New B -Physics*, *JHEP* **03** (2020) 122, [[arXiv:1910.12924](#)]. [Erratum: *JHEP* **04**, 094 (2023)].
- [86] S. Meiser, D. van Dyk, and J. Virto, *Towards a global analysis of the $b \rightarrow c \bar{u} q$ puzzle*, *JHEP* **06** (2025) 019, [[arXiv:2411.09458](#)].

- [87] B. M. Dassing, T. Mannel, and S. Turczyk, *Inclusive semi-leptonic B decays to order $1/m_b^4$* , *JHEP* **03** (2007) 087, [[hep-ph/0611168](#)].
- [88] M. E. Luke and A. V. Manohar, *Reparametrization invariance constraints on heavy particle effective field theories*, *Phys. Lett. B* **286** (1992) 348–354, [[hep-ph/9205228](#)].
- [89] M. Lang, A. Lenz, A. Mohamed, M. L. Piscopo, and A. V. Rusov, *B-meson decay width up to $1/m_b^3$ corrections within and beyond the Standard Model*, *JHEP* **05** (2026) 100, [[arXiv:2512.14635](#)].
- [90] A. Lenz, U. Nierste, and G. Ostermaier, *Penguin diagrams, charmless B decays and the missing charm puzzle*, *Phys. Rev. D* **56** (1997) 7228–7239, [[hep-ph/9706501](#)].
- [91] F. Krinner, A. Lenz, and T. Rauh, *The inclusive decay $b \rightarrow c\bar{c}s$ revisited*, *Nucl. Phys. B* **876** (2013) 31–54, [[arXiv:1305.5390](#)].
- [92] M. Bordone, N. Gubernari, T. Huber, M. Jung, and D. van Dyk, *A puzzle in $\bar{B}_{(s)}^0 \rightarrow D_{(s)}^{(*)+} \{\pi^-, K^-\}$ decays and extraction of the f_s/f_d fragmentation fraction*, *Eur. Phys. J. C* **80** (2020), no. 10 951, [[arXiv:2007.10338](#)].
- [93] T. Huber, S. Kränkl, and X.-Q. Li, *Two-body non-leptonic heavy-to-heavy decays at NNLO in QCD factorization*, *JHEP* **09** (2016) 112, [[arXiv:1606.02888](#)].
- [94] A. Mohamed, *Are Subleading Effects Really Subleading? B-Meson Decays in Mesogenesis*, [arXiv:2511.04858](#).
- [95] A. Khodjamirian and M. Wald, *B-meson decay into a proton and dark antibaryon from QCD light-cone sum rules*, *Phys. Lett. B* **834** (2022) 137434, [[arXiv:2206.11601](#)].
- [96] **Particle Data Group** Collaboration, F. Takahashi et al., *Review of Particle Physics*, *Int. J. Mod. Phys. A* **41** (2026) 2630011.
- [97] **Belle-II, Belle** Collaboration, M. Abumusabh et al., *A search for feebly-interacting particles in B decays with missing energy at Belle*, [arXiv:2601.07104](#).
- [98] **BaBar** Collaboration, J. P. Lees et al., *Search for Evidence of Baryogenesis and Dark Matter in $B^+ \rightarrow \psi_D + p$ Decays at BABAR*, *Phys. Rev. Lett.* **131** (2023), no. 20 201801, [[arXiv:2306.08490](#)].
- [99] J. F. Nieves and P. B. Pal, *Generalized Fierz identities*, *Am. J. Phys.* **72** (2004) 1100–1108, [[hep-ph/0306087](#)].
- [100] T. Mannel, D. Moreno, and A. Pivovarov, *Heavy quark expansion for heavy hadron lifetimes: completing the $1/m_b^3$ corrections*, *JHEP* **08** (2020) 089, [[arXiv:2004.09485](#)].

- [101] A. Lenz, M. L. Piscopo, and A. V. Rusov, *Disintegration of beauty: a precision study*, *JHEP* **01** (2023) 004, [[arXiv:2208.02643](#)].
- [102] M. Egner, M. Fael, A. Lenz, M. L. Piscopo, A. V. Rusov, K. Schönwald, and M. Steinhauser, *Total decay rates of B mesons at NNLO-QCD*, *JHEP* **04** (2025) 106, [[arXiv:2412.14035](#)].
- [103] **Heavy Flavor Averaging Group (HFLAV)** Collaboration, S. Banerjee et al., *Averages of b-hadron, c-hadron, and τ -lepton properties as of 2023*, [arXiv:2411.18639](#).
- [104] D. King, A. Lenz, M. L. Piscopo, T. Rauh, A. V. Rusov, and C. Vlahos, *Revisiting inclusive decay widths of charmed mesons*, *JHEP* **08** (2022) 241, [[arXiv:2109.13219](#)].
- [105] D. Friday, E. Gersabeck, A. Lenz, and M. L. Piscopo, *Charm physics*, [arXiv:2506.15584](#).
- [106] A. Boushmelev and M. Wald, *Higher twist corrections to B-meson decays into a proton and dark antibaryon from QCD light-cone sum rules*, *Phys. Rev. D* **109** (2024), no. 5 055049, [[arXiv:2311.13482](#)].
- [107] D. Espriu, P. Pascual, and R. Tarrach, *Baryon Masses and Chiral Symmetry Breaking*, *Nucl. Phys. B* **214** (1983) 285–298.
- [108] B. L. Ioffe, *Calculation of Baryon Masses in Quantum Chromodynamics*, *Nucl. Phys. B* **188** (1981) 317–341. [Erratum: *Nucl.Phys.B* 191, 591–592 (1981)].
- [109] I. V. Anikin, V. M. Braun, and N. Offen, *Nucleon Form Factors and Distribution Amplitudes in QCD*, *Phys. Rev. D* **88** (2013) 114021, [[arXiv:1310.1375](#)].
- [110] V. M. Braun, A. Lenz, N. Mahnke, and E. Stein, *Light cone sum rules for the nucleon form-factors*, *Phys. Rev. D* **65** (2002) 074011, [[hep-ph/0112085](#)].
- [111] I. I. Balitsky and V. M. Braun, *Evolution Equations for QCD String Operators*, *Nucl. Phys. B* **311** (1989) 541–584.
- [112] A. Lenz, M. Gockeler, T. Kaltenbrunner, and N. Warkentin, *The Nucleon Distribution Amplitudes and their application to nucleon form factors and the $N \rightarrow \Delta$ transition at intermediate values of Q^2* , *Phys. Rev. D* **79** (2009) 093007, [[arXiv:0903.1723](#)].
- [113] A. Khodjamirian, R. Mandal, and T. Mannel, *Inverse moment of the B_s -meson distribution amplitude from QCD sum rule*, *JHEP* **10** (2020) 043, [[arXiv:2008.03935](#)].
- [114] A. G. Grozin and M. Neubert, *Asymptotics of heavy meson form-factors*, *Phys. Rev. D* **55** (1997) 272–290, [[hep-ph/9607366](#)].

- [115] T. Nishikawa and K. Tanaka, *QCD Sum Rules for Quark-Gluon Three-Body Components in the B Meson*, *Nucl. Phys. B* **879** (2014) 110–142, [[arXiv:1109.6786](#)].
- [116] M. Rahimi and M. Wald, *QCD sum rules for parameters of the B-meson distribution amplitudes*, *Phys. Rev. D* **104** (2021), no. 1 016027, [[arXiv:2012.12165](#)].
- [117] C. Bourrely, I. Caprini, and L. Lellouch, *Model-independent description of $B \rightarrow \pi \ell \nu$ decays and a determination of $|V(ub)|$* , *Phys. Rev. D* **79** (2009) 013008, [[arXiv:0807.2722](#)]. [Erratum: *Phys.Rev.D* 82, 099902 (2010)].
- [118] V. M. Braun, A. Lenz, and M. Wittmann, *Nucleon Form Factors in QCD*, *Phys. Rev. D* **73** (2006) 094019, [[hep-ph/0604050](#)].
- [119] G. Hiller, A. Rodríguez-Sánchez, and D. Wendler, *Probing baryon number with missing energy*, [arXiv:2602.15936](#).
- [120] M. A. Shifman, *Wilson Loop in Vacuum Fields*, *Nucl. Phys. B* **173** (1980) 13–31.
- [121] T. Mannel and K. K. Vos, *Reparametrization Invariance and Partial Re-Summations of the Heavy Quark Expansion*, *JHEP* **06** (2018) 115, [[arXiv:1802.09409](#)].
- [122] M. Fael, T. Mannel, and K. Keri Vos, *V_{cb} determination from inclusive $b \rightarrow c$ decays: an alternative method*, *JHEP* **02** (2019) 177, [[arXiv:1812.07472](#)].
- [123] O. V. Tarasov, *Generalized recurrence relations for two loop propagator integrals with arbitrary masses*, *Nucl. Phys. B* **502** (1997) 455–482, [[hep-ph/9703319](#)].
- [124] M. Caffo, H. Czyz, S. Laporta, and E. Remiddi, *The Master differential equations for the two loop sunrise selfmass amplitudes*, *Nuovo Cim. A* **111** (1998) 365–389, [[hep-th/9805118](#)].
- [125] T. Mannel, S. Turczyk, and N. Uraltsev, *Higher Order Power Corrections in Inclusive B Decays*, *JHEP* **11** (2010) 109, [[arXiv:1009.4622](#)].
- [126] T. Mannel, *Higher order $1/m$ corrections at zero recoil*, *Phys. Rev. D* **50** (1994) 428–441, [[hep-ph/9403249](#)].
- [127] M. Kirk, A. Lenz, and T. Rauh, *Dimension-six matrix elements for meson mixing and lifetimes from sum rules*, *JHEP* **12** (2017) 068, [[arXiv:1711.02100](#)]. [Erratum: *JHEP* 06, 162 (2020)].
- [128] G. Finauri and P. Gambino, *The q^2 moments in inclusive semileptonic B decays*, *JHEP* **02** (2024) 206, [[arXiv:2310.20324](#)].
- [129] **Flavour Lattice Averaging Group** Collaboration, S. Aoki et al., *FLAG Review 2019: Flavour Lattice Averaging Group (FLAG)*, *Eur. Phys. J. C* **80** (2020), no. 2 113, [[arXiv:1902.08191](#)].

- [130] **CKMfitter Group** Collaboration, J. Charles, A. Hocker, H. Lacker, S. Laplace, F. R. Le Diberder, J. Malcles, J. Ocariz, M. Pivk, and L. Roos, *CP violation and the CKM matrix: Assessing the impact of the asymmetric B factories*, *Eur. Phys. J. C* **41** (2005), no. 1 1–131, [[hep-ph/0406184](#)].
- [131] A. Lenz, A. Mohamed, and Z. Wüthrich, *Constraining B-mesogenesis models with inclusive and exclusive decays*, *JHEP* **08** (2025) 141, [[arXiv:2412.14947](#)].
- [132] A. Biswas, A. Khodjamirian, and A. Mohamed, *Light-cone sum rules with B-meson distribution amplitudes for the $B \rightarrow p$ form factors in B-mesogenesis models*, *Phys. Rev. D* **113** (2026), no. 11 116035, [[arXiv:2603.01723](#)].
- [133] V. M. Braun, Y. Ji, and A. N. Manashov, *Higher-twist B-meson Distribution Amplitudes in HQET*, *JHEP* **05** (2017) 022, [[arXiv:1703.02446](#)].
- [134] A. Khodjamirian, T. Mannel, and N. Offen, *Form-factors from light-cone sum rules with B-meson distribution amplitudes*, *Phys. Rev. D* **75** (2007) 054013, [[hep-ph/0611193](#)].
- [135] N. Gubernari, A. Kokulu, and D. van Dyk, *$B \rightarrow P$ and $B \rightarrow V$ Form Factors from B-Meson Light-Cone Sum Rules beyond Leading Twist*, *JHEP* **01** (2019) 150, [[arXiv:1811.00983](#)].
- [136] S. Descotes-Genon, A. Khodjamirian, and J. Virto, *Light-cone sum rules for $B \rightarrow K\pi$ form factors and applications to rare decays*, *JHEP* **12** (2019) 083, [[arXiv:1908.02267](#)].
- [137] M. Beneke and T. Feldmann, *Symmetry breaking corrections to heavy to light B meson form-factors at large recoil*, *Nucl. Phys. B* **592** (2001) 3–34, [[hep-ph/0008255](#)].