

PRECISION STUDIES OF SOFT-COLLINEAR QCD DYNAMICS IN
THE PRESENCE OF HEAVY QUARKS

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ABSTRACT

The discovery of the Higgs boson in 2012 marked a milestone in confirming the Standard Model (SM) of particle physics, yet the SM remains incomplete, failing to account for phenomena such as gravity, dark matter, and the matter–antimatter asymmetry in the universe. This motivates precision studies of collider processes, where more accurate theoretical predictions are needed to match the increasing experimental precision. Key challenges arise from Quantum Chromodynamics (QCD), which demands both calculations to high loop-orders in the perturbative regime and a careful separation of perturbative and non-perturbative dynamics. This thesis is split into two parts, each addressing issues in one of these categories.

In the first part of this thesis, we present an all-order analysis of double-logarithmic corrections to the soft-overlap contribution in heavy-to-light transition form factors at large hadronic recoil. We focus on $B_c \rightarrow \eta_c$ transitions in a perturbative non-relativistic setup, treating both the bottom and charm quarks as heavy, with the hierarchy $m_b \gg m_c \gg \Lambda_{\text{QCD}}$. Our diagrammatic analysis identifies two independent sources of double logarithms: soft-gluon effects, described by standard exponential Sudakov factors, and rapidity-ordered soft-quark configurations, which generate a novel set of coupled integral equations. These equations capture the intricate interplay between soft-quark and soft-gluon dynamics at the double-logarithmic level. As an independent consistency check, we employ a bare factorization formula within Soft-Collinear Effective Theory. Although endpoint-divergent convolution integrals prevent its use for resumming logarithmic corrections with renormalization group methods, its structure enables us to derive logarithmic corrections up to the two-loop level. By computing the only unknown contribution, we confirm the correctness of the integral equations to this order. While a closed-form solution of the integral equations remains currently elusive, we provide iterative expressions for the double-logarithmic series and derive the asymptotic behavior of the soft-overlap form factor at infinite recoil, showing that the Sudakov suppression is slightly weakened by the combined effects of soft quarks and soft gluons.

In the second part of this thesis, we study the phase-space integral of the double-emission soft limit of generic QCD amplitudes with massless and massive emitters at an arbitrary angle to each other. This is a necessary ingredient to extend the nested soft-collinear subtraction scheme to cases with massive final states at hadron colliders. We employ integration-by-parts identities and the differential equations method to obtain an analytic expression for the expansion around the small-velocity limit of the massive emitter, which is an important cross-check for the exact calculation with full dependence on the velocity.

Die Entdeckung des Higgs-Bosons im Jahr 2012 war ein Meilenstein für die Bestätigung des Standardmodells (SM) der Teilchenphysik. Dennoch bleibt das SM unvollständig, da es Phänomene wie Gravitation, dunkle Materie und die Asymmetrie zwischen Materie und Antimaterie im Universum nicht erklären kann. Dies motiviert Präzisionsstudien von hochenergetischen Kollisionsprozessen, für die genaue theoretische Vorhersagen erforderlich sind, um der zunehmenden experimentellen Präzision gerecht zu werden. Die größten Herausforderungen ergeben sich dabei aus der Quantenchromodynamik (QCD), die sowohl Berechnungen zu hohen Schleifenordnungen im perturbativen Bereich als auch eine sorgfältige Trennung von perturbativer und nicht-perturbativer Dynamik erfordert. Diese Arbeit ist in zwei Teile gegliedert, die sich jeweils mit Fragen aus einem dieser Bereiche befassen.

Im ersten Teil dieser Arbeit präsentieren wir die doppelt-logarithmischen Korrekturen des Soft-Overlap-Beitrags von schwer-nach-leicht Formfaktoren bei großem hadronischem Rückstoß zu allen Ordnungen der Störungstheorie in QCD. Wir konzentrieren uns dabei auf $B_c \rightarrow \eta_c$ -Übergänge in einem perturbativen nichtrelativistischen Setup, wobei wir sowohl Bottom- als auch Charm-Quarks als schwer behandeln, was mit der Hierarchie $m_b \gg m_c \gg \Lambda_{\text{QCD}}$ realisiert wird. In unserer diagrammatischen Analyse identifizieren wir zwei unabhängige Quellen für doppelte Logarithmen: Soft-Gluon-Effekte, die durch exponentielle Sudakov-Faktoren beschrieben werden, und nach Rapidität geordnete Soft-Quark-Konfigurationen, die einen neuartigen Satz gekoppelter Integralgleichungen erzeugen. Diese Gleichungen erfassen die komplexe Wechselwirkung zwischen der Soft-Quark- und Soft-Gluon-Dynamik auf doppelt-logarithmischer Ebene. Als unabhängiger Konsistenzcheck verwenden wir eine unrenormierte Faktorisierungsformel innerhalb der Soft-Collinear Effective Theory. Obwohl endpunktdivergente Konvolutionen ihre Verwendung für die Summierung logarithmischer Korrekturen mit Renormierungsgruppenmethoden verhindern, ermöglicht ihre Struktur die Berechnung logarithmischer Korrekturen bis zum Zwei-Schleifen-Niveau. Durch die Berechnung des einzigen unbekannten Beitrags bestätigen wir die Richtigkeit der Integralgleichungen bis zu dieser Ordnung. Obwohl eine geschlossene Lösung der Integralgleichungen derzeit noch nicht gefunden werden konnte, liefern wir iterative Ausdrücke für die doppelt logarithmische Reihe und leiten das asymptotische Verhalten des Soft-Overlap-Beitrags bei unendlichem Rückstoß her, wobei wir zeigen, dass die Sudakov-Unterdrückung durch die kombinierten Effekte von weichen Quarks und weichen Gluonen leicht abgeschwächt wird.

Im zweiten Teil dieser Arbeit untersuchen wir das Phasenraumintegral vom Grenzfall zweier weicher Emissionen von generischen QCD-Amplituden mit masselosen und massiven Emitttern in einem beliebigen Winkel zueinander. Dies wird benötigt, um das Nested Soft-Collinear Subtraction Scheme auf Fälle mit massiven Endzuständen an Hadron-Kollidern auszuweiten. Wir verwenden partielle Integration und Differentialgleichungsmethoden, um einen analytischen Ausdruck für die Entwicklung um kleine Geschwindigkeiten des massiven Emitters zu erhalten, was einen wichtigen Konsistenzcheck für die exakte Berechnung mit vollständiger Abhängigkeit von der Geschwindigkeit darstellt.

LIST OF PUBLICATIONS

Parts of this thesis have also appeared in the following publications:

PEER-REVIEWED JOURNAL PUBLICATIONS

- [1] G. Bell, P. Böer, T. Feldmann, D. Horstmann and V. Shtabovenko, *$B_c \rightarrow \eta_c$ form factors at large recoil: Interplay of soft-quark and soft-gluon dynamics*, *JHEP* **09** (2025) 098, [[2412.14149](#)].
- [2] D. Horstmann, K. Melnikov, M.-M. Long and A. Pikelner, *Integral of the double-emission eikonal function for a massive and a massless emitter at an arbitrary angle*, *JHEP* **07** (2025) 138, [[2504.20977](#)].

PROCEEDINGS

- [3] G. Bell, P. Böer, T. Feldmann, D. Horstmann and V. Shtabovenko, *Soft-overlap contribution to $B_c \rightarrow \eta_c$ form factors: diagrammatic resummation of double logarithms*, *PoS RADCOR2023* (2024) 086, [[2309.08410](#)].

IN PREPARATION

- [4] G. Bell, P. Böer, T. Feldmann, D. Horstmann and V. Shtabovenko, *$B_c \rightarrow \eta_c$ form factors at large recoil: SCET analysis and a three-loop check*.

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INTRODUCTION

The discovery of the Higgs boson in 2012 [5, 6] marked a triumph for the Standard Model (SM) of particle physics, which is the theoretical framework that describes the electromagnetic, weak, and strong interactions between elementary particles [7–10]. The role of the Higgs boson in the SM is to give a mass to electro-weak gauge bosons and fermions in a gauge-invariant way via spontaneous symmetry breaking and Yukawa interactions. Its discovery therefore provided critical confirmation of our understanding of the universe at the smallest length scales. The SM has been extraordinarily successful in explaining a vast range of phenomena at widely different energy scales (see [11–13] for an overview). An example at large energy scales are electro-weak precision observables (EWPO) (see e. g. [12]) and at small energy scales the anomalous magnetic moment of the muon [14], both showing excellent agreement between experimental data and theoretical predictions. However, the SM cannot be a complete theory of elementary particles and fundamental forces. It fails to incorporate gravity, leaving the quest for a unified theory that can describe all fundamental forces still open. Additionally, it does not explain other cosmic observations, such as dark matter, dark energy, or the imbalance between matter and antimatter in the universe (see [15, 16] for reviews on Beyond the SM physics). These challenges have driven the search for a theory beyond the SM.

One possibility to further scrutinize the physics of the smallest length scales are earth-based lab experiments and, in particular, collider experiments. An important example is the Large Hadron Collider (LHC) at CERN in Geneva, Switzerland, which accelerates protons at a record collision energy of 13.6 TeV. It hosts two experiments with general purpose detectors, ATLAS and CMS, one that focuses on the forward region particularly interesting for B -physics, LHCb, and one that focuses on heavy-ion physics, ALICE. Another important example is the Belle II experiment at the SuperKEKB accelerator in Tsukuba, Japan, which is an asymmetric e^+e^- collider that operates mainly at the $\Upsilon(4S)$ resonance, decaying primarily into a pair of B mesons, i. e. bound states of heavy b quarks with lighter up or down quarks. Strategies at colliders to search for new particles can be grouped into two categories. The first strategy employs resonant increases of differential cross-sections as a direct evidence for detection. It was e. g. used in the discovery of the Higgs boson, where a resonance in the digamma spectrum could be identified as the Higgs boson. In the absence of experimental signals for such resonances, focus is increasingly shifting to the second strategy: Indirect searches try to find small but statistically relevant discrepancies assumed to originate from virtual quantum fluctuations of heavy unknown particles. It is therefore important to have both precise experimental measurements and theoretical predictions. The expected increase in luminosity at accelerators will lead to a precision of experimental measurements that will be challenging to match by theoretical predictions.

One of the major obstacles to precise theoretical predictions is a detailed understanding of Quantum Chromodynamics (QCD). At large length scales, or equivalently small energy scales, there is a strong attractive force between colored particles which manifests itself in a large strong coupling constant α_s and color confinement, i. e. the fact that one does not observe free colored particles in nature. The large coupling constant implies that this is a regime where it is impossible to use perturbation theory. Instead one has to employ methods such as Lattice QCD, sum rules or a fit to experimental data. On the other hand, asymptotic freedom [17, 18] states that α_s becomes small at large energy scales which allows for a perturbative treatment. Despite being perturbative, α_s is still so large that relatively many orders in α_s are required for precision phenomenology, which comes

with significant challenges. In general, with the current methods any loop order is significantly more involved than the previous one. Nowadays state-of-the-art is next-to-next-to-leading order (NNLO) in QCD for most processes with modest multiplicity and notable exceptions at next-to-next-to-next-to-leading order (N3LO) (see e. g. [19] for an overview).

Due to color confinement one cannot observe quarks and gluons in experiments directly, but always as colorless bound states. At a hadron collider even the initial state is a colorless bound state, namely a proton in case of the LHC. Similarly, at a B factory like Belle II the de-facto initial state of the analysis are B mesons. This implies that even if there is a large energy scale Q involved, aspects of non-perturbative QCD are required for a quantitative description. The separation of perturbative and non-perturbative QCD dynamics is typically organized in the form of an expansion in $\lambda = \Lambda_{\text{QCD}}/Q$, where Λ_{QCD} is the Landau pole of QCD that can be used as a characteristic scale of non-perturbative QCD dynamics. Depending on the value of the large scale Q , power corrections in λ can be sizable. But even if Q is relatively large, power corrections in λ are a necessary ingredient for very precise predictions that will become relevant in the near future. Perhaps the best known example of a separation of scales is the collinear factorization of hadron collider cross sections into parton distribution functions $f_i(x_{1/2})$ and partonic cross sections $d\hat{\sigma}_{ij}(x_1, x_2)$

$$d\sigma = \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) d\hat{\sigma}_{ij}(x_1, x_2) \left(1 + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^n}{Q^n}\right) \right), \quad (0.1)$$

where n depends on the observable and is in general unknown. For $n > 1$ power corrections are currently negligible in the context of LHC phenomenology, but for $n = 1$ they can be relevant for precision observables used, for instance, for the determination of the top-quark mass or the strong coupling constant α_s . Effective Field Theories (EFTs) such as Soft-Collinear Effective Theory (SCET) [20–25] provide a *systematically improvable* framework to study such power corrections. Apart from this, EFTs are a useful tool to resum towers of large logarithms, which occur in any Quantum Field Theory (QFT) whenever there are large scale hierarchies present. At any given order in the power expansion, EFTs separate the dynamics associated to the involved scales in a way that reflects the hierarchies among them. Upon renormalization the objects defined by this separation depend only on their characteristic scale and the renormalization scale. This enables the use of renormalization group (RG) equations that allow to evolve all of these single scale objects to a common scale. This procedure generate evolution kernels that resum logarithms of the different characteristic scales. At leading power the logarithmic accuracy for certain observables obtained by this method has reached an astonishing level with first next-to-next-to-next-to-next-to-leading logarithmic (N4LL) accurate results available [26].

For most EFTs such as Heavy-Quark Effective Theory (HQET) (see [27] for a review) the study of power corrections—including resummation at subleading power—may be technically demanding but without genuinely new conceptual challenges. This is different in SCET where additional difficulties arise at subleading power. In recent years, SCET factorization theorems and the resummation of large logarithmic corrections at subleading power have been studied for a number of processes. Among these are threshold logarithms in Drell-Yan and Higgs production [28–31], e^+e^- event-shape distributions [32–34], the bottom-quark induced $h \rightarrow \gamma\gamma$ and $h \rightarrow gg$ decays [35–39], exclusive and inclusive B -meson decays [40–42] as well as virtual Compton scattering [43]. In SCET energetic degrees of freedom lead to convolution integrals of the functions defined by soft or collinear matrix elements, or hard Wilson coefficients, similar to the structure shown in (0.1). At leading power (LP) these convolution integrals are usually finite, which allows for a separation of scales and a resummation of large logarithms via RG equations. However, it is a generic feature of

SCET at next-to-leading power (NLP) that these convolution integrals diverge at their endpoints. This prevents a renormalization of the separate functions on the integrand level, since the full dependence on the regulator is required to render the endpoint-divergent convolution integrals well-defined. These divergences are arguably the most intricate complication of factorization theorems at subleading power in SCET. For some of the above mentioned processes they have been overcome by means of *additive* refactorization identities [35, 44], which can be used to rearrange the factorization theorem into a form that is free from endpoint divergences. In more complicated cases such as hard exclusive processes these methods appear to be inapplicable.

In the first part of this thesis, we therefore study this problem from a different angle. Namely, we apply diagrammatic resummation techniques to derive the double-logarithmic series of the soft-overlap term of heavy-to-light form factors at large hadronic recoil. The factorization of the power-suppressed soft-overlap term within SCET is known to suffer from endpoint-divergent convolution integrals, which has been an open problem for more than 20 years [45]. It can be viewed as a template case for power corrections to exclusive B decays described at leading power in Λ_{QCD}/m_b by QCD factorization [46, 47]. Given that we currently have no means of estimating power corrections to QCD factorization, it remains challenging to present results with robust uncertainty estimates. In view of the expected reduction of experimental uncertainties it is important to improve our understanding of such power corrections to obtain reliable theoretical predictions for these processes. Since an EFT treatment within SCET is currently not feasible, we resort to diagrammatic techniques to determine the double-logarithmic series of such a process. For concreteness, we study $B_c \rightarrow \eta_c$ transitions in the non-relativistic limit assuming the hierarchy $m_b \gg m_c \gg \Lambda_{\text{QCD}}$ such that the mass of the c quark is small compared to the b quark, but both masses are parametrically larger than Λ_{QCD} , which makes the setup perturbative. We extract double logarithms by placing *physical* cutoffs on loop momenta and find two distinct sources of double-logarithmic contributions: First, on-shell soft gluons give rise to exponential Sudakov factors in the usual way. Second, on-shell soft spectator quarks that are strongly-ordered in rapidity lead to a set of coupled, iterative integral equations that describe the double-logarithmic series. These integral equations constitute the first successful all-order analysis of logarithms for this kind of transitions and place an important constraint on a yet-to-be developed factorization theorem. We perform an independent check of the obtained integral equations by means of a pole cancellation argument based on a bare factorization formula in SCET. While we cannot solve the integral equations in closed form we give a compact form of the coefficients of the double-logarithmic series. In addition, the integral equations allow us to study—for the first time—the asymptotic limit of infinite b quark mass for this kind of transitions and we find that the exponential Sudakov suppression is slightly weakened by the soft-quark contributions.

In the second part of this thesis, we study the phase-space integral of the double-emission soft limit of a generic QCD amplitude with a massless and a massive parton. While the cancellation of infrared singularities is guaranteed by the Kinoshita-Lee-Nauenberg (KLN) theorem [48, 49] for sufficiently inclusive observables, infrared singularities that arise in virtual loop corrections need to be carefully removed by suitable real-radiation contributions. In practice, this amounts to following one of two strategies. The first one, commonly referred to as the slicing method [50], splits the phase space into singular and non-singular pieces with the help of transverse momentum [51] or suitable event-shape variables such as N -jettiness [52]. The singular piece is then described by an appropriate factorization theorem and for the non-singular piece one requires only lower-order amplitudes with higher multiplicity. We focus on the second strategy, subtraction schemes, where all singular limits are subtracted by adding suitable zeros. Subtraction schemes have first been

introduced at NLO by Catani and Seymour in [53]. Since then many realizations have appeared at NNLO and we work in the context of the nested soft-collinear subtraction scheme [54]. We postpone a more detailed description of subtraction schemes to a dedicated section. Notably, at higher loop orders the number and complexity of singular limits increases. At NNLO, one of the most complicated limits is the emission of two soft partons that have a similar small energy. In this case the amplitude factorizes into color-correlated hard matrix elements that are independent of the momenta of the soft emissions in the soft limit, and what we refer to as the double-emission eikonal. The corresponding counter-term is an integral over the double-emission eikonal that has a functional dependence on the angle between the emitters of the soft partons and potentially also their masses in case they are massive. In previous works this integral has been calculated for two massless emitters at an arbitrary angle [55] and for two massive emitters in back-to-back kinematics [56]. To extend the nested soft-collinear subtraction scheme [54] to massive final states at hadron colliders requires the integral of the double-emission eikonal with one massive and one massless emitter at an arbitrary angle. We compute the expansion of this integral around the limit of small velocity of the massive emitter, which serves as an independent check of the corresponding calculation with full dependence on the velocity. The calculation is performed with the help of integration-by-parts identities [57] which significantly reduce the number of loop integrals to only a few master integrals (MIs). These MIs are then solved by means of the differential equations method [58].

The outline of this thesis is as follows. In chapter 1 we introduce a number of methods that are key for the calculations performed in this thesis. In chapter 2 we discuss the setup of the $B_c \rightarrow \eta_c$ calculation. We perform a diagrammatic analysis of the soft-overlap term in the double-logarithmic approximation in chapter 3, first at fixed loop order and then extended to all orders. In sections 4.1 and 4.2 we present the bare factorization formula in SCET and set up a pole-cancellation argument, before we calculate the only missing ingredient to this pole cancellation argument in section 4.3. The derived integral equations are analyzed in chapter 5 and we summarize the findings of the first part of this thesis in chapter 6. In the second part of the thesis, we first discuss the factorization of QCD amplitudes in the presence of two soft emissions and the double-emission eikonal in chapter 7. We calculate the single-emission eikonal in chapter 8 and extend this calculation to the double-emission eikonal expanded around the limit of small velocity of the massive emitter in chapter 9. Finally, we conclude in chapter 9 and give some additional details in the appendices.

We begin by introducing a number of methods that are used in many state-of-the-art phenomenological predictions for collider processes and also provide an important ingredient for the calculations presented in this work. Among them are the method of regions, SCET, integration-by-parts reductions, differential equations and subtraction schemes.

1.1 METHOD OF REGIONS

One method that finds broad applications in modern high-energy physics is the method of regions [59, 60]. It can be used to systematically calculate expansions of dimensionally regulated loop integrals in terms of scale hierarchies present in the integral. We will illustrate how it works by means of two examples.

1.1.1 A simple example

The first example is taken from [61]. The integral

$$I_1 = M^2 \int_0^\infty dk \frac{k}{(k^2 - m^2)(k^2 - M^2)} = \frac{\log \frac{M}{m}}{1 - \frac{m^2}{M^2}} \quad (1.1)$$

resembles a self-energy with vanishing external momentum and two different masses, m and M , as illustrated in Fig. 1.1. The result can be calculated in a straightforward manner, but I_1 is sufficiently simple to serve as a first pedagogical example. Assuming a hierarchy of the masses $m \ll M$, the result (1.1) can be expanded

$$I_1 = \log \frac{M}{m} \left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \mathcal{O}\left(\frac{m^6}{M^6}\right) \right). \quad (1.2)$$

Our goal is to understand how to obtain the expansion (1.2) by working at the integrand level rather than having to calculate the full result first, which can be a demanding task for more complicated integrals.

One possible strategy is to separate the integration domain into regions that have a well-defined scaling with respect to m and M . The disadvantage of this strategy is that it requires calculations with cutoffs, which is inconvenient. We refer to [61] for a more detailed discussion of this strategy. Instead, we expand the integrand in momentum regions given by the present scales without limiting the integration domains. Since this procedure will lead to regions that are separately divergent, we have to introduce a regulator and choose one that resembles dimensional regularization. We work with

$$I_1^{(\varepsilon)} = \mu^{2\varepsilon} M^2 \int_0^\infty dk \frac{k^{1-2\varepsilon}}{(k^2 - m^2)(k^2 - M^2)} = \frac{\log \frac{M}{m}}{1 - \frac{m^2}{M^2}} + \mathcal{O}(\varepsilon), \quad (1.3)$$

where ε is the regulator and μ is a scale introduced to keep the mass dimension of the integral independent of ε . In this example, all relevant momentum regions scale as $k^2 \sim M^2$ and $k^2 \sim$

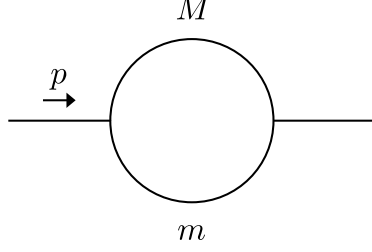


Figure 1.1: A self-energy diagram with two different masses, m and M , and external momentum p . In our example, we set $p^\mu = 0$.

m^2 , which we refer to as the hard and soft region, respectively. Expanding to first order in the momentum scaling of the hard region, $k^2 \sim M^2$, gives

$$I_{1,h}^{(\varepsilon)} = \mu^{2\varepsilon} M^2 \int_0^\infty dk \frac{k^{1-2\varepsilon}}{k^2(k^2 - M^2)} = -\frac{1}{2} \left(\frac{\mu^2}{M^2} \right)^\varepsilon \Gamma(1-\varepsilon)\Gamma(\varepsilon) = -\frac{1}{2\varepsilon} - \log \frac{\mu}{M} + \mathcal{O}(\varepsilon) \quad (1.4)$$

and similarly for the soft region, $k^2 \sim m^2$,

$$I_{1,s}^{(\varepsilon)} = \mu^{2\varepsilon} M^2 \int_0^\infty dk \frac{k^{1-2\varepsilon}}{(k^2 - m^2)M^2} = \frac{1}{2} \left(\frac{\mu^2}{m^2} \right)^\varepsilon \Gamma(1-\varepsilon)\Gamma(\varepsilon) = \frac{1}{2\varepsilon} + \log \frac{\mu}{m} + \mathcal{O}(\varepsilon). \quad (1.5)$$

The sum of both regions

$$I_{1,h}^{(\varepsilon)} + I_{1,s}^{(\varepsilon)} = \log \frac{M}{m} + \mathcal{O}(\varepsilon), \quad (1.6)$$

correctly reproduces the first term of the expansion in (1.2) and, in particular, the divergences of the two regions cancel each other. While the unexpanded integral (1.3) has a non-trivial dependence on two scales, m and M , both the hard and soft region depend only on a single scale through their virtualities μ^2/M^2 and μ^2/m^2 . In the method of regions it is the interplay of the divergences in ε and the virtualities that generates logarithms. It is a straightforward exercise to show that also the higher power terms in (1.2) are obtained correctly if the expansions of the integrands in (1.4) and (1.5) are extended to the required order.

At this point a comment regarding the relevant regions is in order. In general, all regions that lead to a non-vanishing contribution are relevant. In the present example one could, for example, consider an intermediate region $m^2 \ll k^2 \ll M^2$

$$I_{1,m^2 \ll k^2 \ll M^2}^{(\varepsilon)} = \mu^{2\varepsilon} M^2 \int_0^\infty dk \frac{k^{1-2\varepsilon}}{k^2 M^2} = 0, \quad (1.7)$$

which vanishes because the integral is scaleless. It is not always straightforward to find all relevant regions. There is on-going research on algorithmic ways to find regions (see e. g. [62–64] and [65] for a review). However, currently these algorithms do not always find all regions.

1.1.2 A vertex correction with Sudakov logarithms

The second example is a vertex correction, where the external momenta scale such that Sudakov double logarithms are generated. Physically this example represents e. g. a vertex correction of a

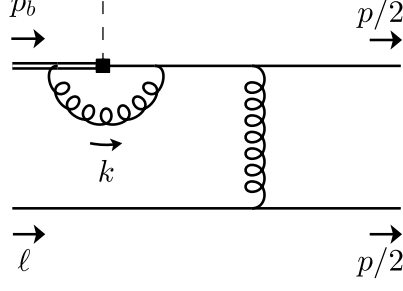


Figure 1.2: A vertex correction diagram typical for heavy-to-light transitions. Double (single) lines describe propagators of heavy (light) quarks. The arrows and their labels indicate momentum flow.

heavy-to-light transition with an energetic final state. The corresponding Feynman diagram can be found in Fig. 1.2, where also all momenta used below are defined. The integral we consider is

$$I_2 = \mu^{2\varepsilon} 2p_b \cdot p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 - 2p_b k + i0^+} \frac{1}{k^2 + i0^+} \frac{1}{(k - p_{hc})^2 + i0^+} . \quad (1.8)$$

In the first propagator we used that the heavy quark is on-shell, $p_b^2 = m_b^2$, and in the last propagator we identified the off-shell momenta, $p - \ell$, entering the loop through the light quark line, cf. Fig. 1.2, with $p_{hc} = p - \ell$. It is well known [66] that the full result is finite and contains dilogarithms, from which it can be non-trivial to extract logarithms that are leading in some kinematical limits. To define the relevant momentum scalings, we introduce a decomposition into light-cone momenta

$$k^\mu = k_- \frac{n^\mu}{2} + k_+ \frac{\bar{n}^\mu}{2} + k_\perp^\mu , \quad (1.9)$$

where $n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$, and $n \cdot k_\perp = \bar{n} \cdot k_\perp = 0$. Then, the scalar product of any two momenta can be written as

$$k \cdot p = \frac{k_+ p_-}{2} + \frac{k_- p_+}{2} + k_\perp \cdot p_\perp . \quad (1.10)$$

The relevant modes of loop and external momenta can be found in Table 1.1 with $Q = m_b$. The physical external momenta scale such that p_b is hard, p is collinear, and ℓ is soft, which implies that $p_{hc} = p - \ell$ is hard-collinear and $p_{b\pm}, p_{hc-} > 0$, $p_{hc+} < 0$. For simplicity we assume that $p_{b+} = p_{b-} = m_b$. The power counting that governs the expansion is given by

$$\lambda \equiv \frac{p_{hc}^2}{p_b^2} \ll 1 . \quad (1.11)$$

In this example, there is no contribution from collinear loop momentum regions. Hence, I_2 can be expanded as

$$I_2 = I_{2,h} + I_{2,hc} + I_{2,s} + \mathcal{O}(\lambda) . \quad (1.12)$$

In the following we calculate each of these regions in turn.

HARD REGION: Expanding (1.8) in the hard momentum region to first order in λ gives

$$I_{2,h} = \mu^{2\varepsilon} 2p_b \cdot p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 - p_{b+} k_- - p_{b-} k_+ + i0^+} \frac{1}{k^2 + i0^+} \frac{1}{k^2 - k_+ p_{hc-} + i0^+} . \quad (1.13)$$

momentum scaling	$(k_-, k_\perp, k_+)$
hard	$(1, 1, 1) Q$
hard-collinear	$(1, \sqrt{\lambda}, \lambda) Q$
collinear	$(1, \lambda, \lambda^2) Q$
soft	$(\lambda, \lambda, \lambda) Q$

Table 1.1: Typical momentum regions if soft-collinear dynamics is involved. Q is a hard scale characteristic to the process under consideration and λ is the power counting parameter.

In this case it is most convenient to perform the integral without transforming the integral measure to light-cone coordinates and splitting the k_- , k_+ and $k_\perp$ integrations from each other. In particular, we can write

$$I_{2,h} = I_2 \Big|_{p_{hc}^\mu = p_{hc-} n^\mu / 2}, \quad (1.14)$$

which implies that the calculation of $I_{2,h}$ amounts to calculating I_2 with $p_{hc}^2 = 0$. One then combines all denominators using Feynman parameters

$$\frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}} = \int_0^1 dx_1 \dots dx_n \delta\left(\sum x_i - 1\right) \frac{\prod x_i^{m_i-1}}{[\sum x_i A_i]^{\sum m_i}} \frac{\Gamma(m_1 + \dots + m_n)}{\Gamma(m_1) \dots \Gamma(m_n)}, \quad (1.15)$$

which can be derived by induction and differentiating

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}, \quad (1.16)$$

with respect to A or B . The result is

$$I_{2,h} = \mu^{2\varepsilon} 2p_b \cdot p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[k^2 - x_1^2 p_b^2 - 2x_1 x_2 p_b p_{hc} + i0^+]^3}. \quad (1.17)$$

It is then straightforward to perform the momentum integration using the standard tadpole formula (see, e.g., [67] for a derivation)

$$\int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{[k^2 + \Delta]^n} = (-1)^n \frac{\Gamma(n - d/2)}{\Gamma(n)} (-\Delta)^{\frac{d}{2} - n}, \quad (1.18)$$

yielding

$$I_{2,h} = -\mu^{2\varepsilon} 2p_b \cdot p_{hc} \Gamma(1 + \varepsilon) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 (x_1^2 p_b^2 + 2x_1 x_2 p_b p_{hc})^{-1-\varepsilon}. \quad (1.19)$$

The remaining integrals of the Feynman parameters evaluate to

$$I_{2,h} = \left(\frac{\mu^2}{p_{b+} p_{b-}} \right)^\varepsilon \frac{\Gamma(1 + \varepsilon)}{2\varepsilon^2} - \left(\frac{\mu^2}{p_{b+} p_{hc-}} \right)^\varepsilon \frac{\Gamma(1 + \varepsilon)}{\varepsilon^2} {}_2F_1 \left(-\varepsilon, \varepsilon; 1 - \varepsilon; 1 - \frac{p_{b-}}{p_{hc-}} \right), \quad (1.20)$$

where ${}_2F_1$ is a hypergeometric function and we write scalar products in terms of their leading power light-cone decomposition. Expanding in ε gives

$$I_{2,h} = e^{-\varepsilon\gamma_E} \left[\left(\frac{\mu^2}{p_{b+}p_{b-}} \right)^\varepsilon \left(\frac{1}{2\varepsilon^2} + \frac{\pi^2}{24} + O(\varepsilon) \right) + \left(\frac{\mu^2}{p_{b+}p_{hc-}} \right)^\varepsilon \left(-\frac{1}{\varepsilon^2} - \frac{\pi^2}{12} + \text{Li}_2 \left(1 - \frac{p_{b-}}{p_{hc-}} \right) + O(\varepsilon) \right) \right], \quad (1.21)$$

where Li_2 is a dilogarithm.

HARD-COLLINEAR REGION: The calculation of the hard-collinear region proceeds in close analogy to the hard region. The expansion of (1.8) in the hard-collinear region gives

$$I_{2,hc} = \mu^{2\varepsilon} 2p_b \cdot p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{-p_{b+}k_- + i0^+} \frac{1}{k^2 + i0^+} \frac{1}{(k - p_{hc})^2 + i0^+}. \quad (1.22)$$

In contrast to the hard region, here, k^2 is power suppressed in the first propagator, which becomes eikonal. Hence, this time we effectively have to calculate

$$I_{2,hc} = \mu^{2\varepsilon} 2p_b \cdot p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{-2p_b k + i0^+} \frac{1}{k^2 + i0^+} \frac{1}{(k - p_{hc})^2 + i0^+} \Big|_{p_b^\mu = p_{b+} \bar{n}^\mu / 2}, \quad (1.23)$$

which implies that we have to set $p_b^2 = 0$. It is most convenient to combine the second and third denominator using Feynman parameters, followed by an application of a related parametric representation

$$\frac{1}{A^m B^n} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^\infty \frac{\lambda^{m-1} d\lambda}{[\lambda A + B]^{n+m}}, \quad (1.24)$$

to arrive at

$$I_{2,hc} = -\mu^{2\varepsilon} 2p_b \cdot p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} 2 \int_0^1 dx \int_0^\infty d\lambda \frac{1}{[k^2 + \bar{x}x p_{hc}^2 + 2x\lambda p_b p_{hc} + i0^+]^3}, \quad (1.25)$$

where $\bar{x} = 1 - x$. The loop integral as well as the Feynman and Georgi parameter integrals can be performed in analogy to the hard region. We obtain

$$I_{2,hc} = - \left(\frac{\mu^2}{-p_{hc-}p_{hc+}} \right)^\varepsilon \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\varepsilon \Gamma(1-2\varepsilon)} = e^{-\varepsilon\gamma_E} \left(\frac{\mu^2}{-p_{hc-}p_{hc+}} \right)^\varepsilon \left(\frac{1}{\varepsilon^2} - \frac{\pi^2}{12} + O(\varepsilon) \right). \quad (1.26)$$

SOFT REGION: For the integral in the soft region we start by rewriting the integral measure in terms of light-cone momenta

$$\int d^d k = \frac{1}{2} \int dk_+ \int dk_- \int d^{d-2} k_\perp. \quad (1.27)$$

Expanding (1.8) in the soft region then leads to

$$I_{2,s} = \mu^{2\varepsilon} p_{b+}p_{hc-} \int dk_+ \int \frac{dk_-}{2\pi i} \int \frac{d^{d-2} k_\perp}{\pi^{\frac{d-2}{2}}} \frac{1}{-p_{b+}k_- - p_{b-}k_+ + i0^+} \times \frac{1}{k_+k_- + k_\perp^2 + i0^+} \frac{1}{p_{hc-}(p_{hc+} - k_+) + i0^+}, \quad (1.28)$$

where we also write k^2 in terms of light-cone momenta. We first perform the k_- -integral as a contour integral. There are no branch cuts and the poles in k_- are

$$k_- \in \left\{ -k_+ + i0^+, -\frac{k_\perp^2}{k_+} - \frac{i0^+}{k_+} \right\}, \quad (1.29)$$

where we used $p_{b+} = p_{b-}$. This implies that $k_+ > 0$, since otherwise all poles are in the upper half-plane and closing the contour around the lower half-plane leads to vanishing integrals. Picking up the pole in the upper half-plane, we arrive at

$$I_{2,s} = \mu^{2\varepsilon} p_{b+} p_{hc-} \int_0^\infty dk_+ \int \frac{d^{d-2}k_\perp}{\pi^{\frac{d-2}{2}}} \frac{1}{-p_{b+} - k_+^2 + k_\perp^2 + i0^+} \frac{1}{p_{hc-} (p_{hc+} - k_+) + i0^+}. \quad (1.30)$$

For the $k_\perp$ integral we can employ (1.18) with $d \rightarrow d - 2$ and omitting the factor of i , because the integral is purely Euclidean

$$I_{2,s} = \mu^{2\varepsilon} \Gamma(\varepsilon) \int_0^\infty dk_+ \frac{(k_+^2 - i0^+)^{-\varepsilon}}{p_{hc+} - k_+ + i0^+}. \quad (1.31)$$

The final integral gives

$$I_{2,s} = - \left(\frac{\mu^2}{p_{hc+}^2} \right)^\varepsilon \Gamma(\varepsilon) \Gamma(2\varepsilon) \Gamma(1 - 2\varepsilon) = e^{-\varepsilon \gamma_E} \left(\frac{\mu^2}{p_{hc+}^2} \right)^\varepsilon \left(-\frac{1}{2\varepsilon^2} - \frac{3\pi^2}{8} + \mathcal{O}(\varepsilon) \right), \quad (1.32)$$

where we used $p_{hc+} < 0$ to write $(-(p_{hc+} + i0^+))^{-2\varepsilon} = (p_{hc+}^2)^{-\varepsilon}$.

RESULT AND DISCUSSION: Combining all regions according to (1.12), we obtain¹

$$I_2 = -\frac{1}{2} \log\left(\frac{p_{b+}}{-p_{hc+}}\right) \log\left(\frac{p_{hc-}^2}{-p_{b+} p_{hc+}}\right) - \frac{\pi^2}{2} + \text{Li}_2\left(1 - \frac{p_{b+}}{p_{hc-}}\right) + \mathcal{O}(\lambda), \quad (1.33)$$

where we used $p_{b+} = p_{b-}$.

We note that the result is finite, as expected. As in the previous example the calculation of single regions is much simpler compared to the calculation of the full integral, because there are fewer scales involved. This can be seen, in particular, in the hard region, which we obtain by setting $p_{hc}^2 = 0$ in the original integral (see (1.14)). Also here the logarithms are generated by the interplay of poles in ε and virtualities in the separate regions. In particular, we notice that double poles in ε lead to double logarithms whereas the single poles in the previous example only lead to single logarithms.

The method of regions can be applied in many other cases and is closely related to effective theories. Examples are the non-relativistic/threshold limit (see e. g. [59, 69, 70]), weak effective theory (see e. g. [71]) and deep expansions in simple hierarchies similar to our first example (see e. g. [72, 73]).

¹ The experienced reader will notice that the result in (1.33) is not reparametrization invariant (RPI) [68], because we used $p_{b-} = p_{b+}$. An expression consistent with RPI can easily be obtained by dropping this constraint in the calculation of the soft region, which results in an extra factor of $(p_{b+}/p_{b-})^\varepsilon$ in $I_{2,s}$.

1.2 SOFT-COLLINEAR EFFECTIVE THEORY

The method of regions allows us to split dimensionally regularized multi-scale integrals into integrals depending only on scales that are parametrically of the same size. The idea of effective field theories is to formulate this scale separation in terms of quantum fields, eventually allowing us to make all-order statements about the factorization of scales. Although it is often not explicitly stated, factorization of scales is a concept frequently used already in classical physics. One example where it is explicit is the multipole expansion in electrodynamics. From afar it is a good approximation to ignore the details of a charge distribution and instead only consider its total charge. Nevertheless, it is possible to systematically take corrections to this approximation into account by considering dipole, quadrupole or even higher moments. There are countless other examples where we intuitively apply factorization of scales. For instance, it is clearly a very good approximation to ignore cosmology when trying to understand the physics of a football. Coming back to QFT there are two important complications related to scale factorization. First, as we will see below, it is technically more challenging to make scale factorization explicit, because quantum fluctuations happen at all scales. Second, in QFT perturbative corrections come with logarithms at every order. As we saw in the previous section, these logarithms depend on hierarchies of scales. As the hierarchies become large, the logarithms grow, which can invalidate the underlying assumption of the perturbative expansion. For concreteness, there are terms that not only come with a factor of the coupling constant α at every order in the perturbative expansion, but also with large logarithms

$$(\alpha(\mu))^n \rightarrow (\alpha(\mu))^n \ln^{bn}(\lambda), \quad (1.34)$$

where λ is a hierarchy of scales and $b = 2$ in case of double logarithms. If λ is sufficiently large $\alpha^n \ln^{bn}(\lambda) \sim \mathcal{O}(1)$ and we have to resum these terms to all orders.

In general, there are two categories of EFTs: bottom-up and top-down. In the former the idea is to use an EFT to systematically parameterize the ignorance of a high-scale theory in terms of fields and symmetries of a well-understood lower scale theory. An example for the bottom-up approach is the Standard Model Effective Theory (SMEFT) (see [74] for a review). In the top-down approach the high-scale theory is known and degrees of freedom with characteristic virtualities at this high scale are integrated out to arrive at a lower scale EFT. This not only provides factorization of scales as described above, but it also gives a field theoretical description more suitable for the kinematic situation at hand compared to the theory including the higher scales. One example is the heavy quark symmetry that becomes manifest in HQET. Commonly used EFTs that belong to the top-down class are the Weak Effective Hamiltonian (see [71, 75, 76] for reviews), HQET, Non-Relativistic QCD (NRQCD, see [77] for a review) and Soft-Collinear Effective Theory (SCET), which we will discuss in detail below along the lines of [61].

SCET [20–25] is the appropriate EFT whenever there are energetic light degrees of freedom that propagate close to the light cone. It was originally invented in the context of heavy-to-light transitions, where, for instance, a heavy b quark decays and deposits its energy into lighter quarks that become energetic. It is also frequently used to describe (hadron) collider processes, since a collider is a machine colliding light, but energetic protons/partons. As a top-down EFT its parent theory is QCD. On a technical level one has to split the QCD fields into modes that correspond to the momentum regions introduced in the section about the method of regions,² cf. Table 1.1,

$$A^\mu(x) \rightarrow A_{hc}^\mu(x) + A_s^\mu(x), \quad \psi(x) \rightarrow \psi_{hc}(x) + \psi_s(x), \quad (1.35)$$

² Here we only discuss SCET-I. The relevant momentum modes are then hard, hard-collinear and soft, but only hard-collinear and soft modes are dynamical degrees of freedom in the EFT.

where $A^\mu(x)$ is a gluon field and $\psi(x)$ is a quark field. Introducing the projectors

$$P_+ = \frac{\not{n}\not{\bar{n}}}{4}, \quad P_+^2 = P_+, \quad P_- = \frac{\not{\bar{n}}\not{n}}{4}, \quad P_-^2 = P_-, \quad (1.36)$$

which obey $P_+P_- = P_-P_+ = 0$, $P_+ + P_- = 1$, and where n and $\bar{n}$ are defined in (1.9), we can further split the collinear quark field

$$\xi = P_+\psi_{hc}, \quad \eta = P_-\psi_{hc}. \quad (1.37)$$

The reason is that in contrast to ψ_{hc} , ξ and η have homogeneous power counting. Specifically, the power counting for all introduced fields is

$$\xi \sim \lambda^{\frac{1}{2}}, \quad \eta \sim \lambda, \quad \psi_s \sim \lambda^{\frac{3}{2}}, \quad \bar{n} \cdot A_{hc} \sim \lambda^0, \quad n \cdot A_{hc} \sim \lambda, \quad A_{\perp hc} \sim \lambda^{\frac{1}{2}}, \quad A_s^\mu \sim \lambda. \quad (1.38)$$

The η and ψ_s fields are power suppressed with respect to ξ , and gluon fields scale as their characteristic momenta. Starting from the QCD Lagrangian

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} (F^a)_{\mu\nu} (F^a)^{\mu\nu}, \quad (1.39)$$

where

$$iD_\mu = i\partial_\mu + g_s A_\mu, \quad ig_s (F^a)^{\mu\nu} t^a = [iD^\mu, iD^\nu], \quad (1.40)$$

are the covariant derivative and field strength tensor, we derive the SCET Lagrangian. Focusing, for now, on the collinear quark sector of the Lagrangian, we begin with

$$\mathcal{L}_{hc} = \bar{\psi}_{hc} i \not{D} \psi_{hc} \quad (1.41)$$

and keep both the collinear and soft fields in the covariant derivative

$$iD_\mu = i\partial_\mu + g_s \left(A_{hc\mu}^a + A_{s\mu}^a \right) t^a. \quad (1.42)$$

Using the projection identities $\not{n}P_+ = 0$ and $\not{\bar{n}}P_- = 0$, (1.41) becomes

$$\mathcal{L}_{hc} = \bar{\xi} \frac{\not{n}}{2} i n \cdot D \xi + \bar{\xi} i \not{D}_\perp \eta + \bar{\eta} i \not{D}_\perp \xi + \bar{\eta} \frac{\not{\bar{n}}}{2} i \bar{n} \cdot D \eta, \quad (1.43)$$

and we notice that all terms have the same power counting in λ according to (1.38). Since the Lagrangian is at most quadratic in η , we can integrate η out, which leads to

$$\mathcal{L}_{hc} = \bar{\xi} \frac{\not{n}}{2} i n \cdot D \xi + \bar{\xi} i \not{D}_\perp \frac{1}{i \bar{n} \cdot D} i \not{D}_\perp \frac{\not{\bar{n}}}{2} \xi. \quad (1.44)$$

In (1.41), we concentrated on the hard-collinear quark field ψ_{hc} since our focus was on the leading-power (LP) Lagrangian, and ψ_s is power-suppressed. Purely soft terms—also with soft quark fields—are, however, not necessarily power suppressed. The reason is that one has to consider the Lagrangian and not the Lagrangian density $\mathcal{L}$, which we usually refer to as the Lagrangian for brevity. The integral measure $d^d x$ present in the Lagrangian scales as $\sim \lambda^{-2}$ in case of hard-collinear or mixed hard-collinear/soft terms, since it is conjugate to the dominating hard-collinear momentum. This implies that all terms in (1.44) are $\sim \lambda^0$ or power suppressed. If there are only soft fields present

in a term, $d^d x$ scales as $\sim \lambda^{-4}$ which compensates the additional power suppression of the fields themselves. Consequently, the purely soft Lagrangian is not power suppressed and reads

$$\mathcal{L}_s = \bar{\psi}_s i \not{D}_s \psi_s - \frac{1}{4} (F_s^a)_{\mu\nu} (F_s^a)^{\mu\nu}, \quad (1.45)$$

where

$$iD_{s\mu} = i\partial_\mu + g_s A_{s\mu}, \quad \text{and} \quad ig_s (F_s^a)^{\mu\nu} t^a = [iD_s^\mu, iD_s^\nu]. \quad (1.46)$$

Putting all pieces together the LP Lagrangian is

$$\mathcal{L}_{\text{SCET}} = \bar{\psi}_s i \not{D}_s \psi_s + \bar{\xi} \frac{\not{n}}{2} \left[in \cdot D + i \not{D}_{hc\perp} \frac{1}{i\bar{n} \cdot D_{hc}} i \not{D}_{hc\perp} \right] \xi - \frac{1}{4} (F_s^a)_{\mu\nu} (F_s^a)^{\mu\nu} - \frac{1}{4} (F_{hc}^a)_{\mu\nu} (F_{hc}^a)^{\mu\nu}, \quad (1.47)$$

where $(F_{hc}^a)_{\mu\nu}$ is the hard-collinear field strength tensor defined below. In contrast to (1.44), in (1.47) we dropped terms in covariant derivatives that are power suppressed. In $D_\perp$ and $\bar{n} \cdot D$ soft fields are dropped, because they are power suppressed. In $n \cdot D$, the situation is different, since $n \cdot A_{hc}$ and $n \cdot A_s$ are of the same power, but in $n \cdot A_s(x)$ parts of the x -dependence must be dropped such that it becomes $n \cdot A_s(x_-)$, as the remaining dependence on x is power suppressed, which is referred to as multipole expansion. To this end, the covariant derivatives and field strength tensors in (1.47) are defined as

$$\begin{aligned} iD_{hc,\mu} &= i\partial_\mu + g_s A_{hc,\mu}, \\ in \cdot D &= in \cdot \partial + g_s n \cdot A_{hc}(x) + g_s n \cdot A_s(x_-), \\ ig_s (F_{hc}^a)^{\mu\nu} t^a &= [iD^\mu, iD^\nu], \end{aligned} \quad (1.48)$$

where D^μ is

$$D^\mu = n \cdot D \frac{\bar{n}^\mu}{2} + \bar{n} \cdot D_{hc} \frac{n^\mu}{2} + D_{hc\perp}^\mu, \quad (1.49)$$

i. e. we keep the soft field in $n \cdot D$ as above. Power corrections to the LP SCET Lagrangian (1.47) are discussed in [24, 25].

SCET inherits gauge transformations as a symmetry from QCD. Gauge transformations

$$U_{s/hc}(x) = \exp \left(i\theta_{s/hc}^a(x) t^a \right), \quad (1.50)$$

can have either soft or hard-collinear scaling in the sense that $\partial\theta_s \sim \lambda\theta_s$ scales soft and similarly for $\partial\theta_{hc}$ has collinear scaling. Under soft gauge transformations the SCET fields transform as

$$\begin{aligned} A_s(x) &\rightarrow U_s(x) A_s(x) U_s^\dagger(x) + \frac{i}{g_s} U_s(x) [\partial, U_s^\dagger(x)], \\ \psi_s(x) &\rightarrow U_s(x) \psi_s(x), \\ A_{hc}(x) &\rightarrow U_s(x_-) A_{hc}(x) U_s^\dagger(x_-), \\ \xi(x) &\rightarrow U_s(x_-) \xi(x), \end{aligned} \quad (1.51)$$

where the soft functions are multipole expanded where appropriate to keep the gauge transformations homogeneous in terms of power counting. The soft fields transform in the same way as for

a QCD gauge transformation. The hard-collinear gauge field A_{hc} transforms covariantly since the additional term with the derivative would be power suppressed except for $n \cdot A_{hc}$ which in (1.47) only appears in a sum with $n \cdot A_s$ and their sum transforms as a usual gauge transformation. Under hard-collinear gauge transformations, on the other hand, the SCET fields transform as

$$\begin{aligned} A_s(x) &\rightarrow A_s(x), \\ \psi_s(x) &\rightarrow \psi_s(x), \\ A_{hc}(x) &\rightarrow U_{hc}(x) A_{hc}(x) U_{hc}^\dagger(x) + \frac{i}{g_s} U_{hc}(x) \left[D_s, U_{hc}^\dagger(x) \right], \\ \xi(x) &\rightarrow U_{hc}(x) \xi(x), \end{aligned} \tag{1.52}$$

where D_s is defined in (1.46). The soft fields do not transform under hard-collinear gauge transformations because it would make them scale as a hard-collinear field. The hard-collinear fields transform as expected with the addition that the derivative in the last term for the gluon field becomes a soft covariant derivative. Again this has the consequence that the sum of A_s and A_{hc} transform as the usual QCD gauge transformation. While the sum of A_s and A_{hc} transforms as the initial QCD field, this is not the case for ξ and ψ_s . The reason is the non-trivial relation of the ξ and ψ_s fields to the QCD field ψ introduced by integrating out η . By straightforward application of the transformation in (1.51) and (1.52) one can check that all covariant derivatives in (1.47) transform covariantly under soft and hard-collinear gauge transformations where appropriate and, hence, the SCET Lagrangian in (1.47) is invariant under these transformations. In particular, we emphasize that the distinct gauge transformation properties of hard-collinear and soft fields allow the corresponding gauge fields to be in separate gauges. We use this notion frequently in a different but related context below.

We will use a simple example to discuss some further aspects of SCET. For $e^+e^- \rightarrow$ dijets mediated by a virtual photon the QCD part of the process is described by the current

$$J^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x). \tag{1.53}$$

Assuming the off-shellness of the virtual photon is hard and the quarks it decays into have (anti-)hard-collinear momenta scaling, the appropriate EFT to describe this process is SCET. Anti-hard-collinear refers to a hard-collinear momentum with the role of n and $\bar{n}$ interchanged and, thus, $\bar{n}$ is the hard direction. In the EFT, the current in (1.53) is represented by (anti-)hard-collinear quark fields that define composite operators which have to be combined with their matching coefficients, commonly referred to as Wilson coefficients. Matching coefficients encode the physics at the hard scale and operators the remaining dynamics at the low scale. One peculiarity of SCET is that (hard-)collinear momenta have small virtualities and are not integrated out, but at the same time they have large momentum components. This has the consequence that derivatives of (hard-)collinear fields along their hard direction are not power suppressed. Hence, in principle any number of derivatives is allowed and we have to consider an infinite tower of operators and corresponding Wilson coefficients. The way out is to consider non-local operators. The idea is that operators with a non-locality along their hard direction and Wilson coefficients depending on this non-locality are equivalent to considering an infinite tower of operators. Applying this idea to the current in (1.53) leads to

$$J^\mu(x) \rightarrow \int ds \int dt C(s, t) \bar{\xi}_{hc}(x + s\bar{n}) \gamma_\perp^\mu \xi_{hc}(x + tn), \tag{1.54}$$

where s and t parameterize the non-locality and $C(s, t)$ is the Wilson coefficient depending on the non-locality. However, the operator in (1.54) has the fundamental problem that it is not gauge invariant. To restore gauge invariance, we introduce Wilson lines

$$[x + s\bar{n}, x] = \mathbf{P} \exp \left[ig_s \int_0^s ds' \bar{n} \cdot A(x + s'\bar{n}) \right], \quad (1.55)$$

where $\mathbf{P}$ is the path ordering operator defined as

$$\mathbf{P} A(x) A(x + s\bar{n}) = A(x + s\bar{n}) A(x) \quad \text{for } s > 0. \quad (1.56)$$

Wilson lines have the convenient feature that they transform as

$$[x + s\bar{n}, x] \rightarrow U(x + s\bar{n}) [x + s\bar{n}, x] U(x), \quad (1.57)$$

under gauge transformations. Assuming that gauge transformations at infinity are trivial in the sense that $U(-\infty\bar{n}) = 1$, it is useful to define

$$W(x) = [x, -\infty\bar{n}] = \mathbf{P} \exp \left[ig_s \int_{-\infty}^0 ds \bar{n} \cdot A(x + s\bar{n}) \right], \quad (1.58)$$

because under a gauge transformations, $W(x)$ transform as

$$W(x) \rightarrow U(x) W(x). \quad (1.59)$$

The transformation behavior in (1.59) allows us to use $W_{hc}^\dagger(x)$

$$W_{hc}(x) = \mathbf{P} \exp \left[ig_s \int_{-\infty}^0 ds \bar{n} \cdot A_{hc}(x + s\bar{n}) \right], \quad (1.60)$$

to cancel the gauge transformation of $\xi(x)$ such that

$$\chi_{hc}(x) = W_{hc}^\dagger(x) \xi(x), \quad (1.61)$$

is gauge invariant. χ_{hc} and its counterpart for gluon fields

$$\mathcal{A}_{hc}^\mu(x) = W_{hc}^\dagger(x) \left(iD_{hc}^\mu W_{hc}(x) \right), \quad (1.62)$$

where the bracket indicates that the covariant derivative acts only on the Wilson line, are referred to as gauge-invariant building blocks, because they can be used to construct gauge invariant non-local operators. Coming back to our operator in (1.53) we have to modify (1.54) such that it is build of gauge-invariant building blocks only

$$J^\mu(x) \rightarrow \int ds \int dt C(s, t) \bar{\chi}_{hc}(x + s\bar{n}) \gamma_\perp^\mu \chi_{hc}^-(x + tn), \quad (1.63)$$

where definitions for anti-hard-collinear fields are analogous to the ones for hard-collinear fields.

At this point the hard-collinear fields interact indirectly with the anti-hard-collinear fields since both of them are coupled to the soft sector through interaction terms in the LP Lagrangian (1.47), which is extended in a straightforward manner to include the anti-hard-collinear fields. There is a field transformation that allows us to decouple the (anti-)hard-collinear fields from the soft fields in the LP Lagrangian. The decoupling transformation is given by [23]

$$\begin{aligned} \xi(x) &= S_n(x_-) \xi^{(0)}(x), \\ A_{hc}^\mu(x) &= S_n(x_-) A_{hc}^{(0)\mu}(x) S_n^\dagger(x_-), \end{aligned} \quad (1.64)$$

where S_n is a soft Wilson line defined as

$$S_n(x) = P \exp \left[ig_s \int_{-\infty}^0 ds \, n \cdot A_s(x + sn) \right]. \quad (1.65)$$

Applying the decoupling transformation (1.64) to the interaction term of the hard-collinear quark field ξ with the covariant derivative containing the soft gauge field yields

$$\bar{\xi} \frac{\not{n}}{2} in \cdot D \xi = \bar{\xi}^{(0)} \frac{\not{n}}{2} in \cdot D_{hc}^{(0)} \xi^{(0)}, \quad (1.66)$$

with D_{hc} defined in (1.48). An analogous relation holds for the interactions between soft and hard-collinear gauge fields introduced through the field strength tensor $(F_{hc}^a)^{\mu\nu}$ in the LP Lagrangian (1.47). Hence, the decoupling transformation removes all interactions between hard-collinear and soft fields from the LP Lagrangian. Consequently, all interactions among soft and hard-collinear fields necessarily cause a power suppression allowing us to formally write down factorization theorems, where the soft and (anti-)hard-collinear dynamics are factorized from each other. To this end, the current in (1.63) becomes after the decoupling transformation

$$J^\mu(x) \rightarrow \int ds \int dt \, C(s, t) \bar{\chi}_{hc}^{(0)}(x + s\bar{n}) S_n(x_-) S_{\bar{n}}^\dagger(x_+) \gamma_\perp^\mu \chi_{hc}^{(0)}(x + tn), \quad (1.67)$$

where we generalized the decoupling transformation and soft Wilson lines for the anti-hard-collinear sector. Since there are both hard-collinear and anti-hard-collinear fields present in this term, x is conjugate to their sum and we have to multipole expand (1.67)

$$J^\mu(x) \rightarrow \int ds \int dt \, C(s, t) \bar{\chi}_{hc}^{(0)}(x_+ + x_\perp + s\bar{n}) S_n(0) S_{\bar{n}}^\dagger(0) \gamma_\perp^\mu \chi_{hc}^{(0)}(x_- + x_\perp + tn) + \dots, \quad (1.68)$$

to arrive at an expression with homogeneous power counting. The Wilson lines do not cancel because one of them is in the n direction and the other one in the $\bar{n}$ direction. Since there is no LP interaction between the soft and (anti-)hard-collinear sectors after the decoupling transformation the soft dynamics is now explicit in the form of soft Wilson lines.

Once a factorization theorem is established, it is possible to systematically determine logarithms to all orders in the expansion in the small coupling constant. This procedure is similar to the extraction of logarithms at fixed order in the method of regions, where the interplay of divergences of single regions with their virtualities generates the logarithms. Translated to the EFT language this means that the functions describing the hard, (anti-)hard-collinear, and soft interactions are renormalized separately. If this is possible, they become single scale objects and renormalization-group equations (RGEs) can be used to evolve all functions to a common scale, resumming all large logarithms. This procedure breaks down if the functions cannot be renormalized separately, which can happen if the convolutions over the non-localities are divergent. We will see an example below in Sec. 4.1.3. In this case the full dependence on the regularization parameter has to be kept until the convolution is performed, which prevents a renormalization. These divergences occur only in some factorization theorems at LP [78] but are a common feature at next-to-leading power (NLP). They are often referred to as endpoint divergences, since they are usually localized at the endpoints of convolution integrals. Endpoint divergences are a long-standing problem in SCET [79] and have been solved only recently for a small number of processes [34–43]. The technique used in these works leverages refactorization identities [35, 44] valid at the endpoint to reshuffle the bare factorization theorem in such a way that the endpoint divergences, which always cancel in the sum of all terms of a factorization theorem, cancel for every term separately, thereby enabling

renormalization at the integrand level. However, to date, this method has evaded applicability for more complicated problems such as hard exclusive processes, which we will discuss in detail below. Instead of relying on a SCET factorization theorem plagued by endpoint divergences, we resort to diagrammatic techniques to understand the systematics of the leading double logarithms to all orders.

1.3 INTEGRATION BY PARTS

Perturbative calculations typically involve large numbers of (multi-)loop integrals. One of the striking benefits of dimensional regularization [80] is that it introduces linear relations among loop integrals, making it possible to express them in a much smaller basis of so-called master integrals (MIs) [57, 81, 82]. This process is commonly referred to as integration-by-parts (IBP) reduction.

Considering a set of integrals with L loop momenta and E independent external momenta, there are $N = L(L + 1)/2 + LE$ linearly independent scalar products involving loop momenta. The first step in an IBP reduction is typically to define topologies, or equivalently sets of denominators, with N linearly independent denominators such that any scalar product involving a loop momentum can be expressed in terms of a denominator. This implies that topologies are closed under differentiation with respect to external or loop momenta, which turns out to be very helpful. Considering for simplicity only a single topology, we can write any integral as

$$B_{v_1, \dots, v_N}(\{s_j, m_j\}, d) = \int \left(\prod_{i=1}^L d^d k_i \right) \prod_{i=1}^N \frac{1}{D_i^{v_i}}, \quad (1.69)$$

where $\{s_j, m_j\}$ are the scalar products of external momenta and masses, and D_i are the denominators.

The next step is to set up linear relations among different integrals. In dimensional regularization loop integrals are invariant under shifts of the loop momenta, which leads to IBP identities

$$0 = \int \left(\prod_{i=1}^L d^d k_i \right) \frac{\partial}{\partial k_j^\mu} \left(\frac{v^\mu}{D_1^{v_1} \dots D_N^{v_N}} \right), \quad (1.70)$$

where v is either a loop momentum or an external momentum. In total this defines $L(L + E)$ linearly independent primary IBP identities among loop integrals belonging to the same topology. Considering infinitesimal Lorentz transformations leads to Lorentz-invariance (LI) identities

$$0 = \sum_{i=1}^E \left(p_i^\nu \frac{\partial}{\partial p_{i\mu}} - p_i^\mu \frac{\partial}{\partial p_{i\nu}} \right) B_{v_1, \dots, v_N}, \quad (1.71)$$

where p_i are the external momenta. LI identities can be helpful, although they do not provide additional information with respect to the IBP identities [83]. One then proceeds by generating finitely many IBP identities according to certain conditions and setting up an ordering criteria, that defines which integrals are eliminated by the above linear relations in favor of more basic ones, namely the MIs. Solving these linear equations produces the reduction of more complicated integrals to MIs. This strategy is referred to as the Laporta algorithm [84]. IBP reduction has been implemented in many public codes, for example, Reduze [85], FIRE [86], Kira [87], LiteRed [88] and Blade [89]. In practice, IBP reductions are an important part of state-of-the-art perturbative calculations, but they often suffer from spurious poles and significant swelling of expressions, resulting in long run times and high demand on RAM. There are many strategies currently under investigation that aim

at improving the efficiency of IBP reductions. Among them are syzygy equations (see e. g. [90]), finite field and interpolation techniques (see e. g. [86]), intersection numbers (see e. g. [91]), and other strategies (see e. g. [87, 92–94]).

1.4 DIFFERENTIAL EQUATIONS

An important way to calculate MIs is the differential equation method [58]. Performing an IBP reduction on the derivative of a MI with respect to (one of) its parameter(s) will result in an expression that depends linearly on the MIs. Combining the derivatives of all MIs, one can write

$$\partial_\beta \vec{I}(d, \beta) = M(d, \beta) \vec{I}(d, \beta), \quad (1.72)$$

where M is a matrix that depends on $d = 4 - 2\varepsilon$ and β is a dimensionless parameter of the MIs. In general M is sufficiently complicated such that it is non-trivial to solve (1.72) directly. In many cases it is possible to find a transformation T

$$\vec{J}(d, \beta) = T(d, \beta) \vec{I}(d, \beta), \quad (1.73)$$

that modifies the differential equation

$$\partial_\beta \vec{J} = T (MT^{-1} - \partial_\beta T^{-1}) \vec{J} = M' \vec{J}, \quad (1.74)$$

such that the ε -dependence factorizes as $M'(d, \beta) = \varepsilon \tilde{M}(\beta)$, commonly referred to as the canonical form [95]

$$\partial_\beta \vec{J}(d, \beta) = \varepsilon \tilde{M}(\beta) \vec{J}(d, \beta), \quad (1.75)$$

which can be solved order-by-order in ε . It is instructive to introduce an example. Given the differential equation matrix

$$M = \begin{pmatrix} -\frac{y}{1-\beta y} & 0 & 0 \\ \frac{1-2\varepsilon}{\beta(1-\beta^2)} & -\frac{1-\beta^2-2\varepsilon(1-\beta y)}{\beta(1-\beta^2)(1-\beta y)} & 0 \\ \frac{1-2\varepsilon}{\beta(1-\beta^2)(1-\beta y)} & \frac{2\beta\varepsilon}{(1-\beta^2)(1-\beta y)} & \frac{2\varepsilon}{\beta(1-\beta y)} \end{pmatrix}, \quad (1.76)$$

which gives the derivative of a set of MIs with respect to β , we can use the transformation

$$T = \begin{pmatrix} \frac{y}{\varepsilon(1-\beta y)} & 0 & 0 \\ \frac{y}{2\varepsilon(1-\beta y)} & \frac{\beta y}{(1-2\varepsilon)(1-\beta y)} & 0 \\ 0 & 0 & -\frac{1}{1-2\varepsilon} \end{pmatrix}, \quad (1.77)$$

to arrive at a differential equation matrix in canonical form

$$M' = T (MT^{-1} - \partial_\beta T^{-1}) = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\beta+1} - \frac{1}{\beta} & \frac{2}{\beta} - \frac{1}{\beta+1} - \frac{1}{\beta-1} & 0 \\ \frac{1}{(\beta+1)y} - \frac{1}{\beta y} & -\frac{1}{(\beta+1)y} + \frac{1}{(\beta-1)y} & \frac{2}{\beta} - \frac{2}{\beta-\frac{1}{y}} \end{pmatrix}. \quad (1.78)$$

We notice that the β dependence of $\tilde{M}$ is of the form

$$\frac{1}{\beta - a_i}, \quad a_i \in \left\{0, 1, -1, \frac{1}{y}\right\}. \quad (1.79)$$

This implies that the solution of the canonical differential equation (1.75) is given in terms of iterated integrals

$$\int_{\beta_0}^{\beta} d\beta_1 \frac{1}{\beta_1 - a_{i_1}} \int_{\beta_0}^{\beta_1} d\beta_2 \frac{1}{\beta_2 - a_{i_2}} \cdots \int_{\beta_0}^{\beta_n} d\beta_n \frac{1}{\beta_n - a_{i_n}}, \quad (1.80)$$

where β_0 is the boundary of the differential equation. The iterated integrals can be written as multiple polylogarithms (MPLs) defined recursively as [96, 97]

$$G(a_{i_1}, a_{i_2}, \dots, a_{i_n}; \beta) = \int_0^{\beta} \frac{d\beta'}{\beta' - a_{i_1}} G(a_{i_2}, \dots, a_{i_n}; \beta'), \quad \text{and} \quad G(\beta) = 1. \quad (1.81)$$

Typically, the solution of the differential equation is straightforward once a transformation to the canonical basis is found. The critical step is to find a transformation to the canonical basis. There are algorithmic ways to find such a transformation [98, 99] that have been implemented in public codes [100–103]. For instance, we used Fuchsia [103] to obtain (1.77). However, in more involved cases one often has to find a transformation by hand. Nevertheless, there is a significant effort to explore algorithms applicable also for elliptic or even more complicated geometries [104–107].

1.5 SUBTRACTION SCHEMES

When computing perturbative corrections to cross sections for any given process there may be ultraviolet (UV) and infrared (IR) divergences. While UV divergences can be removed in an algorithmic fashion by renormalizing the parameters of the Lagrangian under consideration (see e. g. [108] for an introduction to renormalization), IR divergences are more cumbersome. According to the KLN theorem [48, 49] cross sections are only IR finite if they are sufficiently inclusive. For concreteness, this means IR divergences of additional potentially unresolved partons, referred to as real corrections, are needed to cancel IR divergences of virtual loop corrections. In practice, this leads to the problem that for a differential cross section the IR divergences are explicit in virtual corrections, but implicit for real corrections. This problem can be solved efficiently by systematically subtracting the IR divergent parts of the real amplitude as shown by Catani and Seymour in their pioneering work [53] dealing with the NLO case. At NNLO, where the IR divergences of two additional partons have to be analyzed, it is more complicated to find an efficient algorithm. Multiple ways to arrange the subtractions have been worked out in the past. Among them are antenna subtraction [109–112], residue-improved sector decomposition [113–115], the nested soft-collinear subtraction scheme [54, 116, 117] and projection-to-Born subtraction schemes [118]. Another method to obtain differential cross sections are slicing techniques [51, 52, 119, 120], which face other technical challenges mostly associated to large power corrections for small values of the slicing variables.

In this section we will explain the concept of subtraction schemes using a simple example at NLO. In doing so we will largely follow the discussion in [54], where the nested soft-collinear subtraction scheme is introduced, which at NLO is equivalent to the Frixione-Kunszt-Signer (FKS) subtraction scheme [121, 122]. We consider the production of a colorless final state V at a hadron collider. The differential cross section is then given by the convolution

$$d\sigma = \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) d\hat{\sigma}_{ij}(x_1, x_2), \quad (1.82)$$

where $f_{i/j}$ are parton distribution functions (PDFs), $d\hat{\sigma}_{ij}$ are partonic cross sections and the sum runs over all possible incoming partons. For simplicity, we concentrate on the partonic channel

$$q + \bar{q} \rightarrow V. \quad (1.83)$$

When computing the NLO corrections in the strong coupling constant α_s we have to consider the real emission of at most one gluon in the final state. In order to arrive at a finite result both UV and IR divergences must be removed. For this process all UV divergences can be removed by renormalizing α_s . In the $\overline{\text{MS}}$ -scheme this amounts to

$$\frac{\alpha_{s,b}}{2\pi} (\mu_0^2)^\varepsilon S_\varepsilon = \frac{\alpha_s(\mu)}{2\pi} \mu^{2\varepsilon} \left[1 - \frac{\beta_0}{\varepsilon} \frac{\alpha_s(\mu)}{2\pi} + \mathcal{O}(\alpha_s^2) \right], \quad (1.84)$$

where $S_\varepsilon = (4\pi)^\varepsilon e^{-\varepsilon\gamma_E}$, γ_E is the Euler-Mascheroni constant, and

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_f, \quad C_A = N_c, \quad T_R = \frac{1}{2}, \quad (1.85)$$

is the first term in the α_s expansion of the QCD β -function, C_A is the quadratic Casimir of the fundamental representation and T_R is the index of the $SU(3)$ group. IR divergences cancel once additional real radiation is included and collinear divergences are canceled by renormalizing PDFs according to

$$f_{i,b} \rightarrow \left[1 + \frac{\alpha_s(\mu)}{2\pi} \frac{\hat{P}_{ij}^{(0)}}{\varepsilon} + \mathcal{O}(\alpha_s^2) \right] \otimes f_j(\mu), \quad (1.86)$$

where $f_{i,b}$ is the bare PDF, $\hat{P}_{ij}^{(0)}$ are the Altarelli-Parisi splitting functions (cf. e. g. (1.114)) and $\otimes$ indicates a convolution defined by

$$[f \otimes g](z) = \int_0^1 dx dy f(x)g(y)\delta(z - xy). \quad (1.87)$$

We then split the partonic cross section into LO and NLO contributions

$$d\hat{\sigma} \equiv d\hat{\sigma}_{q\bar{q}} = d\hat{\sigma}^{\text{LO}} + d\hat{\sigma}^{\text{NLO}}, \quad (1.88)$$

keeping higher perturbative corrections implicit. Combining real and virtual corrections with (1.86) yields

$$d\hat{\sigma}^{\text{NLO}} = d\sigma^{\text{V}} + d\sigma^{\text{R}} + \frac{\alpha_s(\mu)}{2\pi\varepsilon} \left(\hat{P}_{qq}^{(0)} \otimes d\hat{\sigma}^{\text{LO}} + d\hat{\sigma}^{\text{LO}} \otimes \hat{P}_{qq}^{(0)} \right). \quad (1.89)$$

All three terms in (1.89) are separately divergent and only their sum is finite. Subtraction schemes are in essence strategies to make the divergences in the real radiation explicit. Thus, we will discuss the calculation of $d\sigma^{\text{R}}$ in some detail. To this end, we consider the process

$$q(p_1) + \bar{q}(p_2) \rightarrow V(p_V) + g(k). \quad (1.90)$$

We integrate over the phase space of the additional gluon to be able to add real and virtual corrections in a meaningful way

$$d\sigma^{\text{R}} = \frac{1}{2s} \int [dk] \theta(E_{\text{max}} - k^0) F_{LM}(1, 2, k), \quad (1.91)$$

where $F_{LM}(1, 2, k)$ is defined below, the integral measure for phase space integrals $[dk]$ is defined as

$$[dk] = \frac{d^{d-1}k}{2k^0(2\pi)^{d-1}}, \quad (1.92)$$

and $d = 4 - 2\varepsilon$ is the number of spacetime dimensions as before. The Heaviside function in (1.91) imposes an upper bound $E_{\max}$ on the energy of the emitted gluon. We will see below that with the way we implement the IR limits, the energy would otherwise be unconstrained. The upper bound is crucial to ensure that the integral over the energy k^0 is not scaleless. In other words, we intend to extract only the IR divergences which would be canceled exactly by corresponding UV divergences, unless we impose an upper bound on k^0 . The remaining integrand $F_{LM}(1, 2, k)$ in (1.91) is given by

$$F_{LM}(1, 2, k) = \text{dLips}_V |\mathcal{M}(1, 2, k, V)|^2 \mathcal{F}_{\text{kin}}(1, 2, k, V), \quad (1.93)$$

where dLips_V is the Lorentz-invariant phase space of the uncolored particles V including the momentum conserving $\delta^{(d)}(p_1 + p_2 - k - p_V)$, $\mathcal{M}(1, 2, k, V)$ is the tree-level matrix element of the process (1.90), and $\mathcal{F}_{\text{kin}}(1, 2, k, V)$ is an infrared-safe observable, where the arguments 1 and 2 indicate the dependence on p_1 and p_2 , respectively. It will be convenient to define

$$\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j, \quad (1.94)$$

where $\vec{n}_i$ is the direction of the momentum of the i -th particle in $(d-1)$ -dimensions. The operators

$$S_i A = \lim_{k_i^0 \rightarrow 0} A, \quad C_{ij} A = \lim_{\rho_{ij} \rightarrow 0} A, \quad (1.95)$$

single out leading terms in the soft and collinear limits, respectively. They act on everything that is on their right, here symbolically denoted by A , including the momentum conserving delta function. They are particularly helpful to disentangle the different singular infrared limits allowing us to write

$$\begin{aligned} \int [dk] \theta(E_{\max} - k^0) F_{LM}(1, 2, k) &\equiv \langle F_{LM}(1, 2, k) \rangle = \langle S_k F_{LM}(1, 2, k) \rangle + \langle (I - S_k) F_{LM}(1, 2, k) \rangle \\ &= \langle S_k F_{LM}(1, 2, k) \rangle + \langle (C_{k1} + C_{k2})(I - S_k) F_{LM}(1, 2, k) \rangle + \langle \hat{O}_{\text{NLO}} F_{LM}(1, 2, k) \rangle, \end{aligned} \quad (1.96)$$

where in the last term we used

$$\hat{O}_{\text{NLO}} = (I - C_{k1} - C_{k2})(I - S_k), \quad (1.97)$$

which defines the remaining piece once all singular limits have been subtracted such that this term can be integrated in *four* dimensions.

The operators S_i and C_{ij} defined in (1.95) are our third way of dealing with soft and collinear dynamics. The method of regions allowed us perform the expansion of an integral at the integrand level using momentum modes. In SCET these modes were promoted to quantum fields enabling us to work out all-order factorization theorems valid in regimes with soft-collinear kinematics. On the other hand, the operators S_i and C_{ij} simply define limits acting on, in principle, arbitrary functions of kinematic invariants. It is not part of their definition by which method these limits are obtained. In fact, in many cases the limits are not obtained by means of the method of regions or SCET (see e. g. [114, 123]).

We note that the Heaviside function in (1.96) renders $\langle F_{LM}(1, 2, k) \rangle$ an object that is not Lorentz-invariant. We work in the center-of-mass frame of the incoming partons and all terms in the last line

of (1.96) have to be calculated in the same frame. In the following we will calculate the remaining pieces until we arrive at a form that allows us to explicitly cancel all terms divergent in ε . Before we start doing so, we rearrange infrared limits in the first two terms in the last line of (1.96). More specifically, we write them as

$$\langle (I - C_{k1} - C_{k2}) S_k F_{LM}(1, 2, k) \rangle + \langle (C_{k1} + C_{k2}) F_{LM}(1, 2, k) \rangle. \quad (1.98)$$

The first step in the calculation of the first term in (1.98) is to identify the soft limit

$$S_k F_{LM}(1, 2, k) = \frac{2C_F g_{s,b}}{(k^0)^2} \frac{\rho_{12}}{\rho_{1k} \rho_{2k}} F_{LM}(1, 2), \quad (1.99)$$

where C_F is the quadratic Casimir of the fundamental representation, $g_{s,b}$ is the bare strong coupling constant, $F_{LM}(1, 2)$ is the equivalent of $F_{LM}(1, 2, k)$ for the tree-level process for which we can write

$$\langle F_{LM}(1, 2) \rangle = \int d\text{Lips}_V |\mathcal{M}(1, 2, V)|^2 \mathcal{F}_{\text{kin}}(1, 2, V). \quad (1.100)$$

Acting with the collinear operators on the piece of (1.99) that depends on the relevant angles, yields

$$C_{k1} \frac{\rho_{12}}{\rho_{1k} \rho_{2k}} = \frac{1}{\rho_{1k}} \quad C_{k2} \frac{\rho_{12}}{\rho_{1k} \rho_{2k}} = \frac{1}{\rho_{2k}}. \quad (1.101)$$

The back-to-back kinematics of the incoming quarks in their center-of-mass frame leads to the identities $\rho_{12} = 2$, $\rho_{1k} + \rho_{2k} = 2$ which imply

$$\frac{\rho_{12}}{\rho_{1k} \rho_{2k}} = \frac{1}{\rho_{1k}} + \frac{1}{\rho_{2k}}. \quad (1.102)$$

Consequently, the first term in (1.98) vanishes and for this reason we rearranged the last line of (1.96). We note that this cancellation relies on the specific angular dependence of the soft limit in (1.99) and the back-to-back kinematics of the incoming partons in their center-of-mass frame. At NNLO and in more complicated cases it is preferable to arrange the IR subtraction similar to (1.96) and we refer to [54] for more details. This cancellation leaves us with the calculation of the second term in (1.98). We compute the result for C_{k1} and infer the result for C_{k2} . The collinear limit reads

$$C_{k1} F_{LM}(1, 2, k) = \frac{g_{s,b}^2}{(k^0)^2 \rho_{1k}} (1 - z) P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z}, \quad (1.103)$$

where we defined $z = 1 - k^0/p_1^0$ and the splitting function

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{1-z} - \varepsilon(1-z) \right] = P_{qq}^{(0)}(z) + \varepsilon P_{qq}^{(\varepsilon)}(z). \quad (1.104)$$

Integrating over the angular dependence of (1.103) with the help of (D.61) yields

$$\langle C_{k1} F_{LM}(1, 2, k) \rangle = -\frac{[\alpha_s]}{\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} (2p_1^0)^{-2\varepsilon} \int_{z_{\min}}^1 \frac{dz}{(1-z)^{2\varepsilon}} P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z}, \quad (1.105)$$

where we defined $z_{\min} = 1 - E_{\max}/p_1^0$ and

$$[\alpha_s] = \frac{\alpha_s(\mu)}{2\pi} \frac{\mu^{2\varepsilon} e^{\varepsilon \gamma_E}}{\Gamma(1-\varepsilon)}, \quad (1.106)$$

and $z \cdot 1$ implies zp_1 . The integral in (1.105) still has a soft divergence for $z \rightarrow 1$. We have to make this divergence explicit by splitting it into an ε -pole and plus distributions, to be defined below, to be able to cancel the plus distributions in the Altarelli-Parisi splitting function. To achieve this it is helpful to separate the splitting function $P_{qq}(z)$ into a divergent and a regular piece

$$P_{qq}(z) = \frac{2C_F}{1-z} + P_{qq}^{\text{reg}}(z). \quad (1.107)$$

It is inconvenient to have the lower limit $z_{\min}$ in the z -integral. We can use our freedom to choose $E_{\max}$ such that any $z_{\min}$, i. e. also the one for C_{k2} , obeys $z_{\min} < 0$, which is achieved for $E_{\max} > p_{1/2}^0$. Any $z < 0$ is unphysical and $F_{LM}(z \cdot 1, 2)$ does not have support there. Consequently, we can write

$$\begin{aligned} \int_{z_{\min}}^1 \frac{dz}{(1-z)^{2\varepsilon}} P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z} &= \int_0^1 dz \left[\frac{2C_F}{(1-z)^{1+2\varepsilon}} + (1-z)^{-2\varepsilon} P_{qq}^{\text{reg}}(z) \right] G(z) \\ &= \frac{C_F}{\varepsilon} G(1) + \int_0^1 dz \left[\frac{2C_F}{(1-z)^{1+2\varepsilon}} (G(z) - G(1)) + (1-z)^{-2\varepsilon} P_{qq}^{\text{reg}}(z) G(z) \right], \end{aligned} \quad (1.108)$$

where $G(z) = F_{LM}(z \cdot 1, 2)/z$. The divergence is now explicit, allowing us to expand the integrand in ε . In particular, the previously divergent term becomes

$$\frac{G(z) - G(1)}{(1-z)^{1+2\varepsilon}} = \left[\sum_{n=0}^{\infty} \frac{(-2\varepsilon)^n}{n!} \mathcal{D}_n(z) \right] G(z), \quad (1.109)$$

where $\mathcal{D}_n(z)$ are the plus distributions

$$\mathcal{D}_n(z) = \left[\frac{\ln^n(1-z)}{1-z} \right]_+. \quad (1.110)$$

They are defined to cancel the divergent part of the denominator

$$\int_0^1 dz \left[\frac{1}{1-z} \right]_+ f(z) = \int_0^1 dz \frac{f(z) - f(1)}{1-z}, \quad (1.111)$$

where $f(z)$ is an arbitrary function. Identifying $2p_1^0 = 2p_2^0 = \sqrt{s}$ as the center-of-mass energy of the incoming partons, the final result for the phase space integral over the collinear limit C_{k1} becomes

$$\begin{aligned} \langle C_{k1} F_{LM}(1, 2, k) \rangle &= -\frac{[\alpha_s] s^{-\varepsilon}}{\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \\ &\times \left[-\left(\frac{C_F}{\varepsilon} + \frac{3C_F}{2} \right) \langle F_{LM}(1, 2) \rangle + \int_0^1 dz \mathcal{P}_{qq,R}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} \right\rangle \right], \end{aligned} \quad (1.112)$$

where

$$\mathcal{P}_{qq,R}(z) = \hat{P}_{qq}^{(0)}(z) + \varepsilon \mathcal{P}_{qq,R}^{(\varepsilon)}(z) + \mathcal{O}(\varepsilon^2) \quad (1.113)$$

collects the Altarelli-Parisi splitting function

$$\hat{P}_{qq}^{(0)}(z) = C_F \left[2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z) \right] \quad (1.114)$$

and $\mathcal{P}_{qq,R}^{(\varepsilon)}(z)$ is the splitting function for higher terms in ε

$$\mathcal{P}_{qq,R}^{(\varepsilon)}(z) = C_F [2(1+z) \ln(1-z) - (1-z) - 4\mathcal{D}_1(z)] . \quad (1.115)$$

The calculation of C_{k2} proceeds analogously, enabling us to directly infer the result. This completes the computation of the real radiation contribution, which is given by

$$\begin{aligned} 2s \cdot d\sigma^R = 2[\alpha_s] s^{-\varepsilon} & \left(\frac{C_F}{\varepsilon^2} + \frac{3C_F}{2\varepsilon} \right) \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \langle F_{LM}(1, 2) \rangle + \langle \hat{O}_{\text{NLO}} F_{LM}(1, 2, k) \rangle \\ & - \frac{[\alpha_s] s^{-\varepsilon}}{\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \int_0^1 dz \mathcal{P}_{qq,R}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right\rangle . \end{aligned} \quad (1.116)$$

For the virtual corrections we can arrive at a similar expression, allowing us cancel the divergent terms. It is well-known that the divergent pieces of virtual amplitudes at the 1-loop order factorize into a factor with ε -poles times the tree-level amplitude [124]. For the process we consider, the $\overline{\text{MS}}$ -renormalized virtual cross section can be written as

$$\begin{aligned} 2s \cdot d\sigma^V = \langle F_{LV}(1, 2) \rangle &= \int d\text{Lips}_V 2 \text{Re} \{ \mathcal{M}(1, 2, V) \mathcal{M}_{1\text{-loop}}^*(1, 2, V) \} \mathcal{F}_{\text{kin}}(1, 2, V) \\ &= -2[\alpha_s] \cos(\varepsilon\pi) \left(\frac{C_F}{\varepsilon^2} + \frac{3C_F}{2\varepsilon} \right) s^{-\varepsilon} \langle F_{LM}(1, 2) \rangle + \langle F_{LV}^{\text{fin}}(1, 2) \rangle , \end{aligned} \quad (1.117)$$

where $\langle F_{LV}^{\text{fin}}(1, 2) \rangle$ is the finite remainder. Putting all the pieces in (1.89), (1.116), and (1.117) together we find

$$\begin{aligned} 2s \cdot d\hat{\sigma}^{\text{NLO}} = [\alpha_s] \mathcal{S} \langle F_{LM}(1, 2) \rangle &+ \langle \hat{O}_{\text{NLO}} F_{LM}(1, 2, k) \rangle + \langle F_{LV}^{\text{fin}}(1, 2) \rangle \\ &- \frac{[\alpha_s] s^{-\varepsilon}}{\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \int_0^1 dz \mathcal{P}_{qq,R}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right\rangle \\ &+ \frac{\alpha_s(\mu)}{2\pi\varepsilon} \int_0^1 dz \hat{P}_{qq}^{(0)}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right\rangle , \end{aligned} \quad (1.118)$$

where

$$\mathcal{S} = 2s^{-\varepsilon} \left(\frac{C_F}{\varepsilon^2} + \frac{3C_F}{2\varepsilon} \right) \left(\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} - \cos(\varepsilon\pi) \right) = \frac{2\pi^2 C_F}{3} + \mathcal{O}(\varepsilon) . \quad (1.119)$$

Expanding (1.118) in ε , the final result is

$$\begin{aligned} 2s \cdot d\hat{\sigma}^{\text{NLO}} = & \left\langle F_{LV}^{\text{fin}}(1, 2) + \frac{\alpha_s(\mu)}{2\pi} \left[\frac{2\pi^2 C_F}{3} F_{LM}(1, 2) \right] \right\rangle + \langle \hat{O}_{\text{NLO}} F_{LM}(1, 2, k) \rangle \\ & + \frac{\alpha_s(\mu)}{2\pi} \int_0^1 dz \left[\ln\left(\frac{s}{\mu^2}\right) \hat{P}_{qq}^{(0)}(z) - \mathcal{P}_{qq,R}^{(\varepsilon)}(z) \right] \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right\rangle . \end{aligned} \quad (1.120)$$

In particular, we notice that all ε -poles cancel and that there is no dependence on E_{max} . The final result at NLO consists of the tree-level result, the finite part of the virtual contribution, the real part with singular limits subtracted $\langle \hat{O}_{\text{NLO}} F_{LM}(1, 2, k) \rangle$ and convolutions of splitting functions with the tree-level result.

Similar subtractions can also be performed at NNLO. There the level of complexity is much higher. Even if the 2-loop virtual amplitudes are given, at NNLO one has the so-called real-virtual

terms which require an analysis similar to our example with the modification that the IR limits of 1-loop instead of tree-level amplitudes have to be considered. The most complicated terms is the real-real term with two additional potentially unresolved particles in the final state. This opens up a much richer structure of singular IR limits and the overall efficiency of the approach depends to a large extent on the way subtractions of the IR limits are arranged. One of the IR limits that opens up in real-real contribution is the limit where both additional particles in the final state become soft at the similar rate. The phase space integral of this limit constitutes one of the main parts of this work and we will discuss it in detail starting with chapter 7.

This concludes the discussion of the methods that will be used in the chapters to come.

Part I

$B_c \rightarrow \eta_c$ FORM FACTORS AT LARGE RECOIL

We now turn to the soft-overlap term of heavy-to-light form factors at large hadronic recoil and begin by setting up the framework of the calculation. We employ the non-relativistic approximation in which both the B_c^- and η_c meson are described by their lowest quark-antiquark Fock state with identical velocities. The B_c^- meson consists of a b -quark with momentum $p_b^\mu = m_b v^\mu$ and a light spectator c -antiquark with momentum $\ell^\mu = m_c v^\mu$, where $v^2 = 1$. In the literature, “light” usually implies a massless parton, but in our context “light” refers to the hierarchy $m_c \ll m_b$. We also consider the mass of the c -quark as heavy with respect to Λ_{QCD} ($m_c \gg \Lambda_{\text{QCD}}$). This allows us to treat it as both “light” and non-relativistic at the same time. Another key advantage is that it enables a perturbative treatment [44, 125–127], which is particularly helpful since we aim to understand the underlying systematics of their QCD dynamics. In the non-relativistic limit the η_c -meson is described by a c -quark and a c -antiquark which both have momentum $p_c^\mu = m_c v'^\mu$, where $v'^2 = 1$, and the masses (m_B & m_η) and momenta (p_B & p_η) of the B_c^- and η_c meson are given by the sum of their constituents masses and momenta, respectively. In order to obtain richer structures for consistency checks at different levels of the computation, we formally distinguish the masses of the active c -quark and the spectator c -antiquark such that their momenta are given by $p_1^\mu = m_1 v'^\mu$ and $p_2^\mu = m_2 v'^\mu$, respectively. This also modifies the momentum of the light quark in the B_c^- meson with the effect that from now on $\ell^\mu = m_2 v^\mu$. It is convenient to introduce the ratios $u_0 = m_1/m_\eta$ and $\bar{u}_0 = 1 - u_0 = m_2/m_\eta$, where the physical limit is recovered by $u_0 = \bar{u}_0 = 1/2$. Importantly, both u_0 and $\bar{u}_0$ are treated as $\mathcal{O}(1)$ quantities which implies that the mass hierarchies translate to $m_b \gg m_1 \approx m_2 \gg \Lambda_{\text{QCD}}$. We stress that this setup is not intended to provide a precise phenomenological prediction for the $B_c \rightarrow \eta_c$ transition. It rather allows us to understand conceptual aspects related to factorization in a well-defined perturbative setup. The spinor degrees-of-freedom of the pseudoscalar mesons are described by the projectors

$$\mathcal{P}_B = \frac{if_B m_B}{4N_c} (1 + \not{v}) \gamma_5, \quad \mathcal{P}_\eta = \frac{if_\eta m_\eta}{4N_c} \gamma_5 (1 + \not{v}'), \quad (2.1)$$

which can be determined by requiring that they fulfill the relevant equations of motion (EOM) and that their normalization recovers the standard definition of the decay constants¹

$$\langle 0 | \bar{q}_2 \gamma^\mu \gamma_5 b | B_c(p_B) \rangle = if_B p_B^\mu, \quad \langle \eta_c(p_\eta) | \bar{q}_2 \gamma^\mu \gamma_5 q_1 | 0 \rangle = -if_\eta p_\eta^\mu. \quad (2.2)$$

The momentum transfer of the form factor is

$$q^2 = (m_B v - m_\eta v')^2 \approx m_B^2 - 2m_B m_\eta v \cdot v'. \quad (2.3)$$

Hence, in the rest frame of the B_c^- meson the boost of the η_c meson is given by

$$\gamma \equiv v \cdot v' \approx \frac{m_B^2 - q^2}{2m_B m_\eta}, \quad (2.4)$$

¹ We do not distinguish between the decay constants defined in QCD and HQET, since they differ only by single logarithms, which can be neglected in the double-logarithmic approximation.

and its maximal value in the physical region ($q^2 \geq 0$) is $\gamma = m_B/(2m_\eta)$. At large recoil ($\gamma \gg 1$), which is our kinematic region of interest, it is convenient to decompose v and v' in terms of the light-cone coordinates introduced in (1.9)

$$v^\mu = \frac{1}{2} (n^\mu + \bar{n}^\mu) , \quad v'^\mu \approx \gamma n^\mu + \frac{1}{4\gamma} \bar{n}^\mu , \quad (2.5)$$

where in the latter the approximation refers to the fact that with this definition, $v'^2 = 1$ holds exactly while (2.4) matches only up to power corrections in $\lambda \equiv m_\eta/m_B$.

We define the soft-overlap term of the heavy-to-light form factor as

$$F(\gamma) \equiv \frac{1}{2E_\eta} \langle \eta_c(p_\eta) | (\bar{q}_1 \Gamma b)(0) | B_c(p_B) \rangle , \quad \text{with} \quad \Gamma = \frac{\not{n} \not{\bar{n}}}{4} . \quad (2.6)$$

This definition projects onto the non-factorizable contribution. We postpone a more detailed discussion of this definition to section 4.1, where we consider the factorization of such form factors in SCET.

In this chapter, we will diagrammatically work out the all-order systematics of the double-logarithmic contributions to the $B_c^- \rightarrow \eta_c$ form factor at large recoil. We will start by describing our method to extract the double-logarithmic contributions in some detail by calculating the subset of diagrams that are relevant up to the two-loop order. There are two kinds of configurations that generate a double-logarithmic contribution: rapidity-ordered, on-shell soft-quarks, and on-shell soft-gluons. We will disentangle the Dirac structure associated with pure soft-quark configurations to all orders and discuss how to dress soft-quarks with soft-gluons. In addition, we will analyze all previously omitted diagrams and show that they do not contribute at the double-logarithmic level.

3.1 FIXED-ORDER ANALYSIS

At large recoil, the transition from initial to final state comes with a parametrically large increase in energy for the spectator antiquark. Hence, there has to be at least one interaction between the active and spectator quarks to transfer momentum to the spectator quark such that it becomes parametrically as energetic as the active quark in the η_c meson. At leading order in the perturbative expansion in the strong coupling constant α_s , there are two Feynman diagrams that realize this momentum transfer, shown in Fig. 3.1. For instance, T_2 evaluates to

$$T_2 = \frac{\text{Tr} \left[\Gamma \mathcal{P}_B i t^a \gamma^\mu \mathcal{P}_\eta i t^a \gamma^\nu i (\not{p}_\eta - \not{\ell} + u_0 m_\eta) \right]}{(p_\eta - \ell)^2 - u_0^2 m_\eta^2} G_{\mu\nu}(p_2 - \ell), \quad (3.1)$$

where $G_{\mu\nu}^{ab}$ is the gluon propagator and the trace acts on both spinor and color space. Note that here and in the following we do not consider the factor $1/2E_\eta$ in (2.6) to be part of a diagram, while it is part of the form factor. Adding the contributions from both diagrams and expanding to leading power, we obtain at tree-level

$$F^{(0)}(\gamma) = \xi_0 \frac{2 + \bar{u}_0}{\bar{u}_0^3}, \quad \text{with} \quad \xi_0 = \frac{\alpha_s C_F}{4\pi} \frac{\pi^2 f_B f_\eta m_B}{N_c E_\eta^2 m_\eta}, \quad (3.2)$$

where we defined $F(\gamma) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n F^{(n)}(\gamma) + \mathcal{O}(m_\eta/m_b)$.

It will be convenient to work in light-cone gauge (LCG) $\bar{n} \cdot A = 0$ in which the gluon propagator reads

$$G^{\mu\nu}(k) = \frac{-i}{k^2} \left[g^{\mu\nu} - \frac{\bar{n}^\mu k^\nu + \bar{n}^\nu k^\mu}{\bar{n} \cdot k} \right] = \frac{-i}{k^2} \left[g_\perp^{\mu\nu} - \frac{n \cdot k}{\bar{n} \cdot k} \bar{n}^\mu \bar{n}^\nu - \frac{\bar{n}^\mu k_\perp^\nu + \bar{n}^\nu k_\perp^\mu}{\bar{n} \cdot k} \right]. \quad (3.3)$$

The reason is that in LCG interactions of energetic gluons with heavy quarks are power suppressed. For instance, for the part of diagram T_1 that describes the active quark line and the B meson projector $\mathcal{P}_B$ we obtain

$$\frac{\Gamma(\not{p}_b + \not{\ell} - \not{p}_2 + m_b) \gamma^\mu \mathcal{P}_B}{(p_b + \ell - p_2)^2 - m_b^2} \approx \frac{\Gamma v^\mu \mathcal{P}_B}{-v \cdot p_2}, \quad (3.4)$$

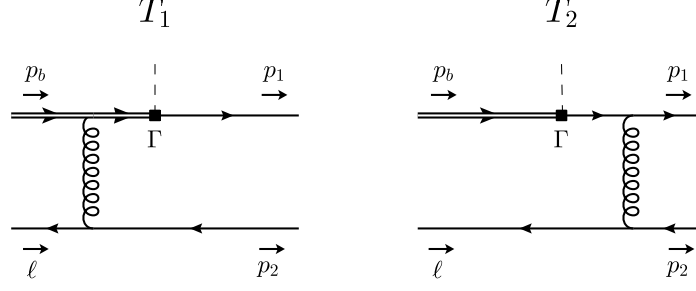


Figure 3.1: Diagrams T_1 and T_2 contributing at leading order to the soft-overlap form factor $F(\gamma)$. The double line represents the heavy b -quark and the solid lines are the active light quark (upper line) and the spectator antiquark (lower line) with masses m_1 and m_2 , respectively. The black square marked with Γ that connects the double and solid line in the active quark line indicates the operator insertion defined in (2.6).

which yields a subleading contribution upon contracting with the gluon propagator $G_{\mu\nu}(p_2 - \ell)$ (3.3) and taking the trace, because $v^\mu g_{\perp,\mu\nu} = 0$ and the other terms in $G_{\mu\nu}(p_2 - \ell)$ (3.3) are power suppressed. This reasoning can be extended to higher orders and significantly constrains the class of diagrams that has to be considered. In the EFT language this statement corresponds to the fact that energetic gluons couple to the heavy b -quark via eikonal Wilson-line interactions because of the specific projection properties of the soft-overlap form factor encoded in Γ . At tree level, this has the consequence that in LCG there is no contribution from T_1 to (3.2) and one only needs to consider T_2 .

3.1.1 Extracting leading logarithms efficiently

At NLO, logarithms start to appear. We are interested in understanding their coefficients to all orders in the perturbative series and focus on terms that come with two large logarithms $L \equiv \ln(2\gamma)$ per order in α_s , which we refer to as the double-logarithmic approximation. Since EFT methods are currently not capable of resumming the leading double logarithms (LDLs) to all orders for the considered process (see e. g. [128]), let alone subleading logarithms, we employ an alternative approach that focuses on the LDLs. The general idea is to understand what kind of logarithmic structures appear in order to facilitate the construction of an endpoint-finite EFT factorization theorem that must reflect these logarithmic structures. The purpose of this section is to motivate and introduce this alternative approach based on the examples studied in Sec. 1.1.

Before we discuss the more involved example dealing with the vertex correction and Sudakov logarithms also relevant for the present calculation, we turn to the simpler example with a large hierarchy between two masses (cf. Sec. 1.1.1). We start with two important observations. The first one is that we obtain the correct logarithm for the example discussed in Sec. 1.1.1 by combining the leading singularities in ε of all regions with their virtualities

$$I_1^\varepsilon = -\frac{1}{2\varepsilon} \left(\frac{\mu^2}{M^2} \right)^\varepsilon + \frac{1}{2\varepsilon} \left(\frac{\mu^2}{m^2} \right)^\varepsilon = \ln \left(\frac{M}{m} \right) + \mathcal{O}(\varepsilon). \quad (3.5)$$

The second observation is that we can write the contributions to (3.5) as integrals of the integrand in the intermediate region $m^2 \ll k^2 \ll M^2$ (see Eq. (1.7)) with appropriate cutoffs

$$-\frac{1}{2\varepsilon} \left(\frac{\mu^2}{M^2} \right)^\varepsilon = \mu^{2\varepsilon} \int_0^M dk k^{-1-2\varepsilon}, \quad \frac{1}{2\varepsilon} \left(\frac{\mu^2}{m^2} \right)^\varepsilon = \mu^{2\varepsilon} \int_m^\infty dk k^{-1-2\varepsilon}. \quad (3.6)$$

Combining both integrals gives

$$\begin{aligned}
I_1^\varepsilon &= \mu^{2\varepsilon} \int_0^M dk k^{-1-2\varepsilon} + \mu^{2\varepsilon} \int_m^\infty dk k^{-1-2\varepsilon} \\
&= \mu^{2\varepsilon} \int_0^\infty dk k^{-1-2\varepsilon} + \mu^{2\varepsilon} \int_m^M dk k^{-1-2\varepsilon} \\
&= \int_m^M \frac{dk}{k} + O(\varepsilon) = \ln\left(\frac{M}{m}\right) + O(\varepsilon),
\end{aligned} \tag{3.7}$$

where in the first step we rearranged the integral regions. In the second step we used that the first integral in the second line is scaleless and the second integral is finite and can be expanded in ε on the integrand level. Effectively, we derived that it is possible to combine the integrand of an intermediate region, that is scaleless in a method of regions analysis, with *physical cutoffs* to obtain the leading logarithms. The reason is that the full information with leading poles and virtualities characteristic for the regions is an overly complicated way of calculating the leading logarithm. Due to pole cancellation there is only one degree of freedom, which is the overall prefactor, that can, as it turns out, be extracted from the intermediate region. Here we compare the performance of a separate method of regions analysis for every perturbative order to the method we describe. To illustrate and understand this method further, we now translate this notion to the second example in Sec. 1.1.2, the vertex correction with Sudakov logarithms.

Similar to (3.5) we can extract the LDLs in this example by writing

$$I_2 \simeq -\frac{1}{2\varepsilon^2} \left(\frac{\mu^2}{\Lambda_0^2}\right)^\varepsilon + \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{\Lambda_0\Lambda_1}\right)^\varepsilon - \frac{1}{2\varepsilon^2} \left(\frac{\mu^2}{\Lambda_1^2}\right)^\varepsilon = -\frac{1}{2} \ln^2\left(\frac{\Lambda_0}{\Lambda_1}\right) + O(\varepsilon), \tag{3.8}$$

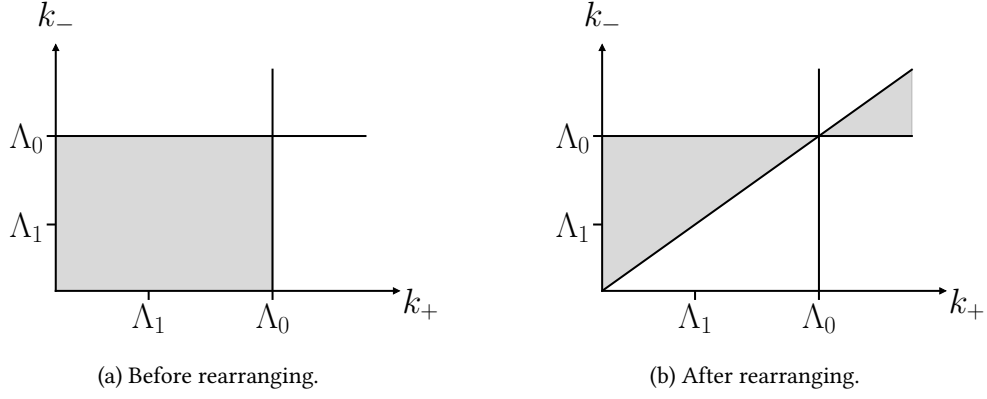
where we use “ $\simeq$ ” to indicate the double-logarithmic approximation. We will make frequent use of this notation in the following. Further, we identified all light-cone momenta that have power counting λ^p with Λ_p , which is the only relevant information at the double-logarithmic level. Whereas in the previous example the integral was one-fold from the beginning, here, the situation is more complicated. In order to obtain a double-logarithmic result one has to find a way to arrive at a two-fold integral of the form

$$\int \frac{dx}{x} \int \frac{dy}{y}, \tag{3.9}$$

with an appropriate integral domain. In a problem with soft-collinear kinematics it is natural that the role of x and y is taken up by the k_+ and k_- components of a loop momentum k . This implies that the $k_\perp$ components have to be integrated out at this point and one has to start from an intermediate region that renders at least two denominators eikonal, one for k_+ and one for k_- . All regions that fulfill this constraint are viable and have to be checked for possible contributions. The only region that renders the integral (1.8) eikonal is

$$\begin{aligned}
k^2 - 2p_b k &\stackrel{!}{=} -p_{b+} k_- &\implies & k_+ \ll k_- \\
(k - p_{hc})^2 &\stackrel{!}{=} -p_{hc-} k_+ &\implies & k_- \ll p_{hc-}, \quad p_{hc+} \ll k_+.
\end{aligned} \tag{3.10}$$

In the second equation in (3.10) k_- cannot dominate, because p_{hc-} is present. Since there are only two denominators available that could give the eikonal factors, this denominator must provide the

Figure 3.2: Integration domain of $I_{2,h,DL}$.

factor k_+ and the upper equation has to provide the required k_- . Hence, we have to consider the integral

$$I_{2,\text{eikonal}} = \mu^{2\varepsilon} 2p_b p_{hc} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{-p_b k_- + i0^+} \frac{1}{k^2 + i0^+} \frac{1}{-k_+ p_{hc} + i0^+}. \quad (3.11)$$

We transform the integral measure to light-cone coordinates and perform the $\perp$ integration

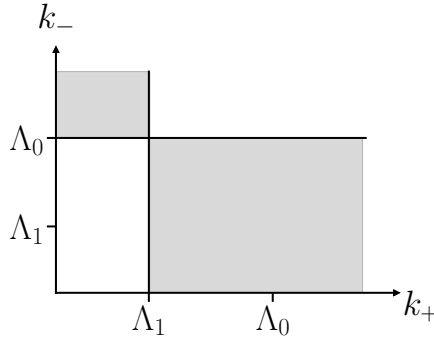
$$\int \frac{d^{d-2} k_\perp}{\pi^{\frac{d-2}{2}}} \frac{1}{k_+ k_- + k_\perp^2 + i0^+} = -\Gamma(\varepsilon) (-k_+ k_- - i0^+)^{-\varepsilon}, \quad (3.12)$$

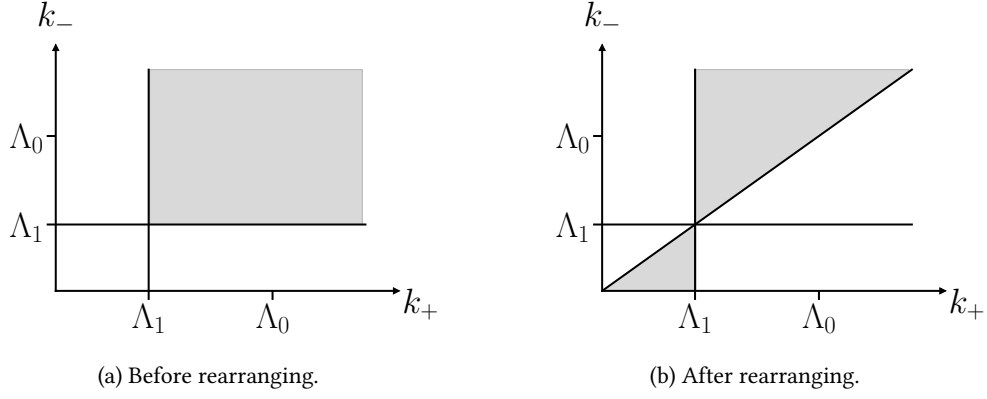
to arrive at

$$I_{2,\text{eikonal}} = -\frac{\Gamma(\varepsilon)}{2\pi i} \int dk_+ \int dk_- \frac{1}{k_- - i0^+} \frac{1}{k_+ - i0^+} (-k_+ k_- - i0^+)^{-\varepsilon}. \quad (3.13)$$

To resolve the analytic structure of the remaining two-fold integral we perform a contour integral over k_+ . There is a pole at $k_+ = i0^+$ and a branch cut for $-k_+ k_- - i0^+ < 0$, which implies that $k_- > 0$, because otherwise both pole and branch cut are in the upper half-plane. We choose to integrate over the lower half-plane and, hence, have to integrate over the branch cut $k_+ + i0^+/k_- > 0$. The result is

$$I_{2,\text{eikonal}} = -\frac{1}{\Gamma(1-\varepsilon)} \int_0^\infty dk_+ \int_0^\infty dk_- \frac{1}{k_+} \frac{1}{k_-} (k_+ k_-)^{-\varepsilon}. \quad (3.14)$$

Figure 3.3: Integration domain of $I_{2,hc,DL}$.

Figure 3.4: Integration domain of $I_{2,s,DL}$.

In the following we drop the factor of $1/\Gamma(1-\varepsilon)$, since higher terms in ε are not relevant at the double-logarithmic level. It is immediately clear that the integral in (3.14) is scaleless. However, this was clear from the beginning, since otherwise this region would have contributed in the method of regions analysis. Our aim was to determine the integrand that we have to supply with appropriate limits in order to reproduce the double logarithm in (3.8). For the hard and soft regions we can write

$$\begin{aligned}
 I_{2,h,DL} &= -\frac{1}{2\varepsilon^2} \left(\frac{\mu^2}{\Lambda_0^2} \right)^\varepsilon = -\frac{1}{2} \mu^{2\varepsilon} \int_0^{\Lambda_0} \frac{dk_+}{k_+^{1+\varepsilon}} \int_0^{\Lambda_0} \frac{dk_-}{k_-^{1+\varepsilon}}, \\
 I_{2,s,DL} &= -\frac{1}{2\varepsilon^2} \left(\frac{\mu^2}{\Lambda_1^2} \right)^\varepsilon = -\frac{1}{2} \mu^{2\varepsilon} \int_{\Lambda_1}^\infty \frac{dk_+}{k_+^{1+\varepsilon}} \int_{\Lambda_1}^\infty \frac{dk_-}{k_-^{1+\varepsilon}},
 \end{aligned} \tag{3.15}$$

and for the hard-collinear region we split the integral in two to have the same prefactor

$$I_{2,hc,DL} = \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{\Lambda_0 \Lambda_1} \right)^\varepsilon = -\frac{1}{2} \mu^{2\varepsilon} \int_{\Lambda_1}^\infty \frac{dk_+}{k_+^{1+\varepsilon}} \int_0^{\Lambda_0} \frac{dk_-}{k_-^{1+\varepsilon}} - \frac{1}{2} \mu^{2\varepsilon} \int_0^{\Lambda_1} \frac{dk_+}{k_+^{1+\varepsilon}} \int_{\Lambda_0}^\infty \frac{dk_-}{k_-^{1+\varepsilon}}, \tag{3.16}$$

as in the hard and soft regions. For interactions with the hard line that has isotropic power counting, it turns out to be convenient to rearrange parts of the soft and hard integrals. The reason is that for isotropic interactions there is no preferred direction leading to a factor of two and it is more convenient to translate this factor of two to the integral domain rather than having it in the integrand. For instance, for the soft integral the rearrangement is

$$\begin{aligned}
 I_{2,s,DL} &= -\frac{1}{2} \mu^{2\varepsilon} \int_{\Lambda_1}^\infty \frac{dk_+}{k_+^{1+\varepsilon}} \int_{\Lambda_1}^\infty \frac{dk_-}{k_-^{1+\varepsilon}} = -\frac{1}{2} \mu^{2\varepsilon} \int_{\Lambda_1}^\infty \frac{dk_+}{k_+^{1+\varepsilon}} \int_{\Lambda_1}^{k_+} \frac{dk_-}{k_-^{1+\varepsilon}} - \frac{1}{2} \mu^{2\varepsilon} \int_{\Lambda_1}^\infty \frac{dk_+}{k_+^{1+\varepsilon}} \int_{k_+}^\infty \frac{dk_-}{k_-^{1+\varepsilon}} \\
 &= -\frac{1}{2} \mu^{2\varepsilon} \int_0^{\Lambda_1} \frac{dk_+}{k_+^{1+\varepsilon}} \int_0^{k_+} \frac{dk_-}{k_-^{1+\varepsilon}} - \frac{1}{2} \mu^{2\varepsilon} \int_{\Lambda_1}^\infty \frac{dk_+}{k_+^{1+\varepsilon}} \int_{k_+}^\infty \frac{dk_-}{k_-^{1+\varepsilon}},
 \end{aligned} \tag{3.17}$$

where we used the symmetry of the integrand under $k_+ \leftrightarrow k_-$ and that the integrand leads to scaleless integrals unless provided with appropriate limits. It is useful to compare the integral domains graphically, which is possible because all integrals have the same integrand. Illustrations of the hard, hard-collinear and soft integrals can be found in Figs. 3.2, 3.3, and 3.4, respectively. Adding up the integrals in Figs. 3.2b, 3.3, 3.4b leads to the integration domain depicted in Fig. 3.5a. The integration domain in Fig. 3.5a can, in turn, be split into one scaleless part (see Fig. 3.5b) and the final result in Fig. 3.5c, which reads

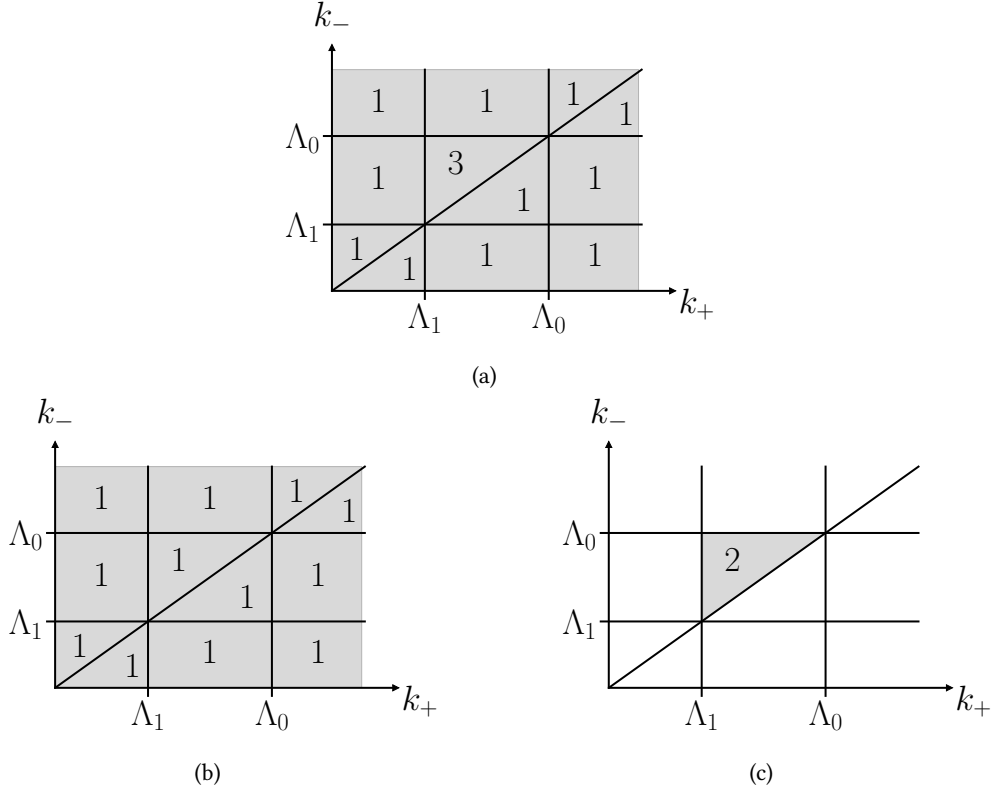


Figure 3.5: The sum of $I_{2,h,DL}$, $I_{2,hc,DL}$, $I_{2,s,DL}$ in (3.5a) can be split into (3.5b) and (3.5c). Numbers in the diagrams indicate multiplicative factors of the integrand (3.14) in this region.

$$I_2 \simeq - \int_{\Lambda_1}^{\Lambda_0} \frac{dk_+}{k_+} \int_{k_+}^{\Lambda_0} \frac{dk_-}{k_-} = -\frac{1}{2} \ln^2 \left(\frac{\Lambda_0}{\Lambda_1} \right), \quad (3.18)$$

where we dropped the ε dependence, because the integral is finite. The important insight is that we could have arrived at this result if we had simply applied the constraints (3.10) to the two-fold eikonal integral in (3.14). This is a useful shortcut, because it implies that it is sufficient to determine the prefactor of the integrand of an eikonal region—in other examples it might be a few eikonal regions—rather than having to calculate the leading poles in ε for a number of regions. In particular, we will see that for almost all diagrams there is only one eikonal region that has to be considered, while calculating the leading poles becomes impractical at higher loop orders. Our strategy to obtain the leading double logarithms can be summarized as follows:

1. Given a particular Feynman diagram with an arbitrary number of loops, one has to find the on-shell regions that render a sufficient number of denominators eikonal, that is at least one for k_{n+} and one for k_{n-} for every loop momentum k_n .
2. Instead of performing an analysis of the analytic structure for every loop momentum, it turns out that it is sufficient to effectively set the propagator, that defines the on-shellness of a given region, on-shell by replacing

$$\frac{1}{k_n^2 - m^2 + i0^+} \rightarrow -2\pi i \delta(k_{n+} k_{n-} + k_{n\perp}^2 - m^2), \quad (3.19)$$

where m is the mass of the particle that goes on-shell. For $m = 0$ we recover our previous result and it is straightforward to show that the same is true if we repeat the analysis of the analytic structure taking m into account.

3. After using the on-shell constraint (3.19) the double-logarithmic terms can be isolated by considering only terms that have the form

$$C \prod_n \int \frac{dk_{n+}}{k_{n+}} \int \frac{dk_{n-}}{k_{n-}}, \quad (3.20)$$

where C is a constant independent of the loop momenta.

4. The final result for the double-logarithmic approximation is given by combining (3.20) with the limits corresponding to the region that lead to the eikonal structure in the first place.

We further illustrate the use of this strategy by explicitly calculating all relevant diagrams at NLO and NNLO before we start to analyze the all-order systematics of the double-logarithmic approximation.

3.1.2 Next-to-leading order

We now return to the calculation of the form factor (2.6). At NLO, the result is known to be [44, 126]

$$F^{(1)}(\gamma) \simeq \xi_0 L^2 \left(C_F \frac{1 + 2\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{2\bar{u}_0^3} \right), \quad \text{with} \quad L \equiv \ln(2\gamma), \quad (3.21)$$

which has two noteworthy properties. First, in contrast to a Sudakov exponentiation there is a C_A term. Second, even the C_F term does not factorise to the tree-level result (3.2). We will see that both of these features are a consequence of on-shell soft-quark configurations that arise along with standard on-shell soft-gluon configurations. These two configurations, and combinations of them, are the only regions that produce double-logarithmic contributions according to the strategy laid out in the previous section. In the following, we will define and address each of them in turn.

SOFT-QUARK CORRECTIONS: We define a soft-quark to be a configuration where a spectator quark becomes on-shell and at the same time its momentum has a scaling that renders the gluon propagators connecting the active and spectator quark line eikonal. The diagram D_1 in Fig. 3.6 is one of the diagrams where a soft-quark gives a double-logarithmic contribution. Concentrating on the denominators we can write

$$D_1 = \int \frac{d^4k}{(2\pi)^4} \frac{\text{Numerator}[D_1]}{[k^2 - m_2^2] (\ell - k)^2 (k - p_2)^2 [(p_\eta - k)^2 - m_1^2] [(p_\eta - \ell)^2 - m_1^2]}, \quad (3.22)$$

where $\text{Numerator}[D_1]$ collects all terms that may contribute to the numerator and, in particular, also the denominators specific to LCG that are eikonal irrespective of the considered momentum region. Once we realize that we can effectively use the on-shell constraint to perform the $\perp$ integration, the $i0^+$ prescription becomes obsolete and we keep it implicit from here on. The region

$$\begin{aligned} l_- &\ll k_- \ll p_{2-}, \\ l_+ &\gg k_+ \gg p_{2+}, \end{aligned} \quad (3.23)$$

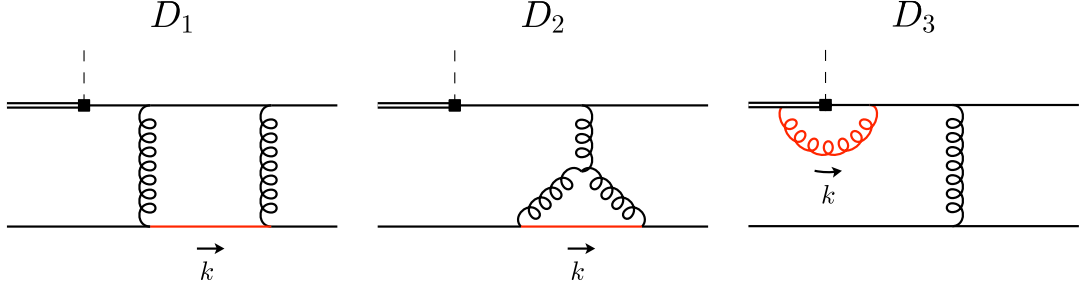


Figure 3.6: All diagrams contributing at NLO in hybrid gauge. Red propagators indicate on-shell soft quark or gluon configurations.

makes the propagators in (3.22) eikonal

$$D_1 \approx \int \frac{d^4 k}{(2\pi)^4} \frac{\text{Numerator}[D_1]}{[k^2 - m_2^2] (-\ell_+ k_-) (-k_+ p_{2-}) (-k_+ p_{\eta-}) (-\ell_+ p_{\eta-})}, \quad (3.24)$$

and we refer to it as a soft-quark contribution, because it is both *eikonal* and it is the *spectator quark* that goes on-shell. It should be noted that the ordering in (3.23) is not soft in the traditional, isotropic sense (see Table 1.1). In terms of ordering it is between a soft and a collinear mode. In case of a massive particle going on-shell, implementing the on-shell constraint (3.19) and integrating over the $\perp$ components generates an additional constraint on the remaining two-fold integral in form of a Heaviside function

$$\theta(k_+ k_- - m_2^2). \quad (3.25)$$

In terms of analytic structure this corresponds to the fact that one has to integrate over the branch cut $k_+ + i0^+ / k_- > m_2^2 / k_-$. Consequently, the remaining eikonal two-fold integral becomes

$$\int_{\ell_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{p_{2+}}^{\ell_+} \frac{dk_+}{k_+} \theta(k_+ k_- - m_2^2) = \frac{1}{2} \ln^2 \left(\frac{\ell_+ p_{2-}}{m_2^2} \right) = \frac{1}{2} L^2. \quad (3.26)$$

In order to arrive at the integral (3.26) the numerator has to provide exactly one power of k_+ , because the denominators in (3.24) provide two inverse powers of k_+ . Extracting these terms yields

$$\text{Numerator}[D_1] \simeq 384i\pi\alpha_s C_F \gamma^4 m_\eta^6 (1 + \bar{u}_0) k_+ \xi_0. \quad (3.27)$$

There is no other region in D_1 that provides double logarithms. Putting all pieces together, the contribution of diagram D_1 to the form factor is

$$F_{D_1}^{(1)}(\gamma) \simeq \xi_0 L^2 \left(3C_F \frac{1 + \bar{u}_0}{\bar{u}_0^3} \right). \quad (3.28)$$

The analysis of diagram D_2 (see Fig. 3.6) proceeds along the same lines. Implementing the region (3.23), we obtain

$$D_2 \approx \int \frac{d^4 k}{(2\pi)^4} \frac{\text{Numerator}[D_2]}{[k^2 - m_2^2] (-\ell_+ k_-) (-k_+ p_{2-}) (-\ell_+ p_{2-}) (-\ell_+ p_{\eta-})}. \quad (3.29)$$

In contrast to diagram D_1 the numerator should be independent of k_+ and k_- , since (3.29) is already in the form of (3.26). The relevant terms of the numerator are

$$\text{Numerator}[D_2] \simeq -64i\pi\alpha_s C_A \gamma^4 m_\eta^7 \bar{u}_0^2 \xi_0. \quad (3.30)$$

Hence, the contribution of diagram D_2 to the form factor is

$$F_{D_2}^{(1)}(\gamma) \simeq \xi_0 L^2 \left(-\frac{C_A}{2\bar{u}_0^3} \right). \quad (3.31)$$

We note in passing that even though a soft quark does not have a momentum scaling of an isotropic soft mode, it behaves as expected from a soft quark in the sense that in contrast to a soft gluon its Dirac structure does not factorize to the tree-level Dirac structure, which can be deduced from the fact the polynomial structure in $\bar{u}_0$ is different compared to the one at tree level (3.2).

In LCG, D_1 and D_2 are the only diagrams that produce double logarithms via soft-quark configurations at NLO. At the level of the discussion in this section it is a priori not clear why for instance the crossed version of D_1 does not yield double logarithms. We postpone a systematic analysis why such kinds of diagrams do not contribute double logarithms to Sec. 3.2.2.

SOFT-GLUON CORRECTIONS: We define an on-shell soft-gluon configuration in analogy to a soft-quark configuration, except that here it is a gluon that goes on-shell. In hybrid gauge—to be defined below—there is effectively only one diagram at NLO that gives the leading double logarithms originating from a soft-gluon configuration, which is D_3 in Fig. 3.6. Here, the on-shell soft gluon depicted in red is in Feynman gauge, while the other gluon is in LCG. We note that this is a non-standard gauge choice, but postpone a more detailed discussion to the next paragraph. Identifying $p_{hc} = p_\eta - \ell$ the denominator structure in D_3 is exactly the same as in the second example in Sec. 3.1.1, see (1.8). Hence, the appropriate eikonal region is the one in (3.10). In this region the Dirac structure factorizes from the tree-level contribution and we can write

$$D_3 \approx T_2 \times 4\pi\alpha_s C_F \int \frac{d^4k}{(2\pi)^4} \frac{(-2im_b p_{\eta-})}{[k^2] (-m_b k_-)(-k_+ p_{\eta-})}, \quad (3.32)$$

where T_2 is the tree-level diagram in Fig. 3.1 corresponding to D_3 . Writing out the integral limits explicitly instead of using the Λ_p notation, the eikonal integral reads

$$\int_{\ell_+}^{p_{\eta-}} \frac{dk_-}{k_-} \int_{\ell_+}^{k_-} \frac{dk_+}{k_+} = \frac{1}{2} \ln^2 \left(\frac{p_{\eta-}}{\ell_+} \right) = \frac{1}{2} \ln^2 \left(\frac{2\gamma}{\bar{u}_0} \right) \simeq \frac{1}{2} L^2, \quad (3.33)$$

and the corresponding contribution to the form factor is

$$F_{D_3}^{(1)}(\gamma) \simeq \xi_0 L^2 \left(-C_F \frac{2 + \bar{u}_0}{\bar{u}_0^3} \right) = -C_F L^2 F^{(0)}(\gamma). \quad (3.34)$$

With all NLO results at hand, we note that sum of the contributions from individual diagrams in (3.28), (3.31), and (3.34) reproduces the well-known total result at NLO in (3.21) and, in particular, that both the non-factorization to the tree-level result and the C_A term mentioned in the beginning of this section originate exclusively from soft-quark configurations. Before we continue with the explicit analysis at NNLO we comment on our gauge.

Until considering soft-gluon configurations we were working exclusively in LCG. For the soft-quark contributions this is convenient, because, for instance, in Feynman gauge also the equivalents of diagrams $D_{1/2}$ with gluon attachments to the b quark would contribute. For soft gluons, on the other hand, it turns out to be most convenient to use Feynman gauge. Hence, from now on, we work in what we refer to as hybrid gauge, which is defined as follows. Gluons that have scaling resembling a hard-collinear mode are treated in LCG (e.g. the gluons in Fig. 3.6 depicted in black). If

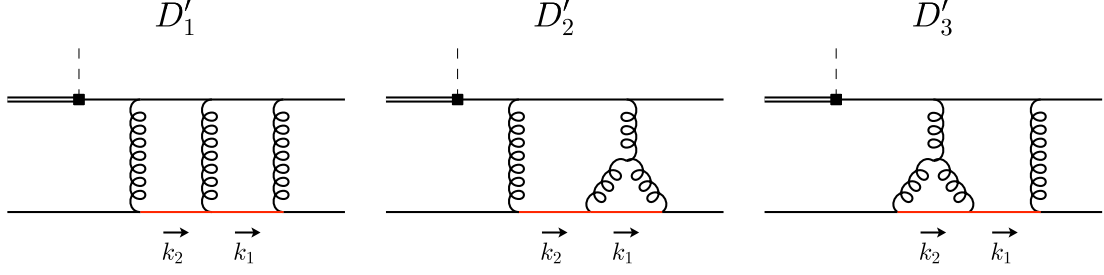


Figure 3.7: Two-loop diagrams that capture the leading logarithmic corrections from double soft-quark insertions in light-cone gauge.

there are on-shell soft-gluon configurations, they are treated in Feynman gauge (e.g. the red gluon in Fig. 3.6). In diagrams $D_{1/2}$ all gluons were treated in LCG as they should be in hybrid gauge. Thus, the above discussion of these diagrams does not need to be modified. The only other diagram that contributes at NLO in hybrid gauge is diagram D_3 . Here, the on-shell soft gluon depicted in red is in Feynman gauge, while the other gluon is in LCG.¹ Having gluons with different gauge choices in one calculation is non-standard in QCD, but natural in SCET. We explicitly checked at NLO and NNLO that the result in hybrid gauge matches the result in pure Feynman gauge. We stress that the consistency of our gauge choice can be understood by drawing an analogy to SCET, where gauge fields with different momentum scalings transform independently under gauge transformations (cf. Sec. 1.2).

3.1.3 Next-to-next-to-leading order

Turning to the double-logarithmic contributions at NNLO we will see that there is a new effect: Mixing between soft-quark and soft-gluon configurations. Before we turn to these mixing effects we discuss the two-loop equivalents of the soft-quark and soft-gluon diagrams at NLO.

DOUBLE SOFT-QUARK CORRECTIONS: In hybrid gauge there are three soft-quark diagrams that contribute to the double-logarithmic approximation at NNLO, shown in Fig. 3.7. The gluon propagators connecting the active and spectator quark become eikonal if the on-shell soft-quark momenta scale as

$$\begin{aligned} l_- &\ll k_{2-} \ll k_{1-} \ll p_{2-}, \\ l_+ &\gg k_{2+} \gg k_{1+} \gg p_{2+}. \end{aligned} \quad (3.35)$$

The eikonal propagators in combination with the physical cutoff generated by (3.35) lead to the double-logarithmic integral

$$\int_{\ell_-}^{p_{2-}} \frac{dk_{2-}}{k_{2-}} \int_{k_{2-}}^{p_{2-}} \frac{dk_{1-}}{k_{1-}} \int_{m_2^2/k_{2-}}^{\ell_+} \frac{dk_{2+}}{k_{2+}} \int_{m_2^2/k_{1-}}^{k_{2+}} \frac{dk_{1+}}{k_{1+}} = \frac{1}{12} L^4. \quad (3.36)$$

We note that these nested integrals are similar to those describing the double-logarithmic series of electron-muon backward scattering [78, 129]. To understand the total contribution of a specific

¹ In this discussion it appears as if the color coding in Feynman diagrams refers to the gauge of a particular gluon. It should be noted, however, that particles drawn in red refer to on-shell configurations. Only in case of at most a single on-shell soft gluon per diagram it is the case that a black gluon is automatically in LCG.

diagram we have to analyze the denominator and numerator of a given diagram in more detail. For diagram D'_1 the denominator structure is given by

$$D'_1 \approx \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{\text{Numerator}[D'_1]}{[k_1^2 - m_2^2][k_2^2 - m_2^2](\ell_+ k_{2-})(k_{2+} k_{1-})(k_{1+} p_{2-})(k_{1+} p_{\eta-})(k_{2+} p_{\eta-})(\ell_+ p_{\eta-})}, \quad (3.37)$$

which implies that the numerator has to give a factor $k_{1+} k_{2+}$ to arrive at the double-logarithmic integral in (3.36). Isolating the relevant terms, we obtain

$$\text{Numerator}[D'_1] \simeq -8192\pi^2 \alpha_s^2 C_F^2 \gamma^5 m_\eta^7 (1 + \bar{u}_0) k_{1+} k_{2+} \xi_0, \quad (3.38)$$

and plugging this into (3.37) yields

$$F_{D'_1}^{(2)}(\gamma) \simeq \xi_0 L^4 \left(\frac{4}{3} C_F^2 \frac{1 + \bar{u}_0}{\bar{u}_0^3} \right). \quad (3.39)$$

We note that while there is a picture emerging regarding the denominators of these kinds of diagrams, it is non-trivial to develop a systematic understand of the relevant terms in the numerator and we postpone a detailed analysis to the next section. Diagrams D'_2 and D'_3 are also eikonalized by the strong rapidity ordering in (3.35). Repeating the steps that lead to (3.39) gives

$$F_{D'_2}^{(2)}(\gamma) \simeq \xi_0 L^4 \left(-\frac{C_A C_F}{6\bar{u}_0^3} \right), \quad F_{D'_3}^{(2)}(\gamma) \simeq \xi_0 L^4 \left(-C_A C_F \frac{1 + \bar{u}_0}{6\bar{u}_0^3} \right), \quad (3.40)$$

and, consequently, the total contribution of pure soft-quark diagrams at NNLO is

$$F_{\text{soft quarks}}^{(2)}(\gamma) \simeq \xi_0 L^4 \left(\frac{4}{3} C_F^2 \frac{1 + \bar{u}_0}{\bar{u}_0^3} - C_A C_F \frac{2 + \bar{u}_0}{6\bar{u}_0^3} \right). \quad (3.41)$$

DOUBLE SOFT-GLUON CORRECTIONS: At NLO we saw that the only relevant soft gluon diagram factorizes to the only tree-level diagram that contributes in hybrid gauge, T_2 (cf. Fig. 3.1). The pure soft gluon contributions behave as expected: The one-loop soft gluon logarithm exponentiates to all-orders and generates a multiplicative, exponential Sudakov factor proportional to the tree-level form factor $F^{(0)}(\gamma)$. We will not discuss them further here. Expanding the Sudakov factor to two-loops we obtain

$$F_{\text{soft gluons}}^{(2)}(\gamma) \simeq \xi_0 L^4 \left(\frac{1}{2} C_F^2 \frac{2 + \bar{u}_0}{\bar{u}_0^3} \right) = \frac{1}{2} C_F^2 L^4 F^{(0)}(\gamma). \quad (3.42)$$

MIXED SOFT-QUARK AND SOFT-GLUON CORRECTIONS: The soft-quark diagrams that we discussed at NLO also receive corrections from soft-gluon attachments. Because there are more options for the gluon to attach, this comes with an increased number of diagrams. Since there is an additional complication for the soft-gluon corrections to the box diagram D_1 , we begin with a discussion of the corrections to the non-Abelian diagram D_2 . All diagrams relevant in hybrid gauge and their respective phase spaces that render the diagrams eikonal and generate the logarithms are depicted in Fig. 3.8. The first diagram, D'_4 , is the equivalent of the NLO soft-gluon correction to the tree-level diagram T_2 , except the Born of this diagram is the non-Abelian diagram D_2 . This is straightforward to understand since the heavy-to-light vertex is not affected by the loop correction. In other words, its input momenta remain unchanged and the Dirac structure factorizes as expected.

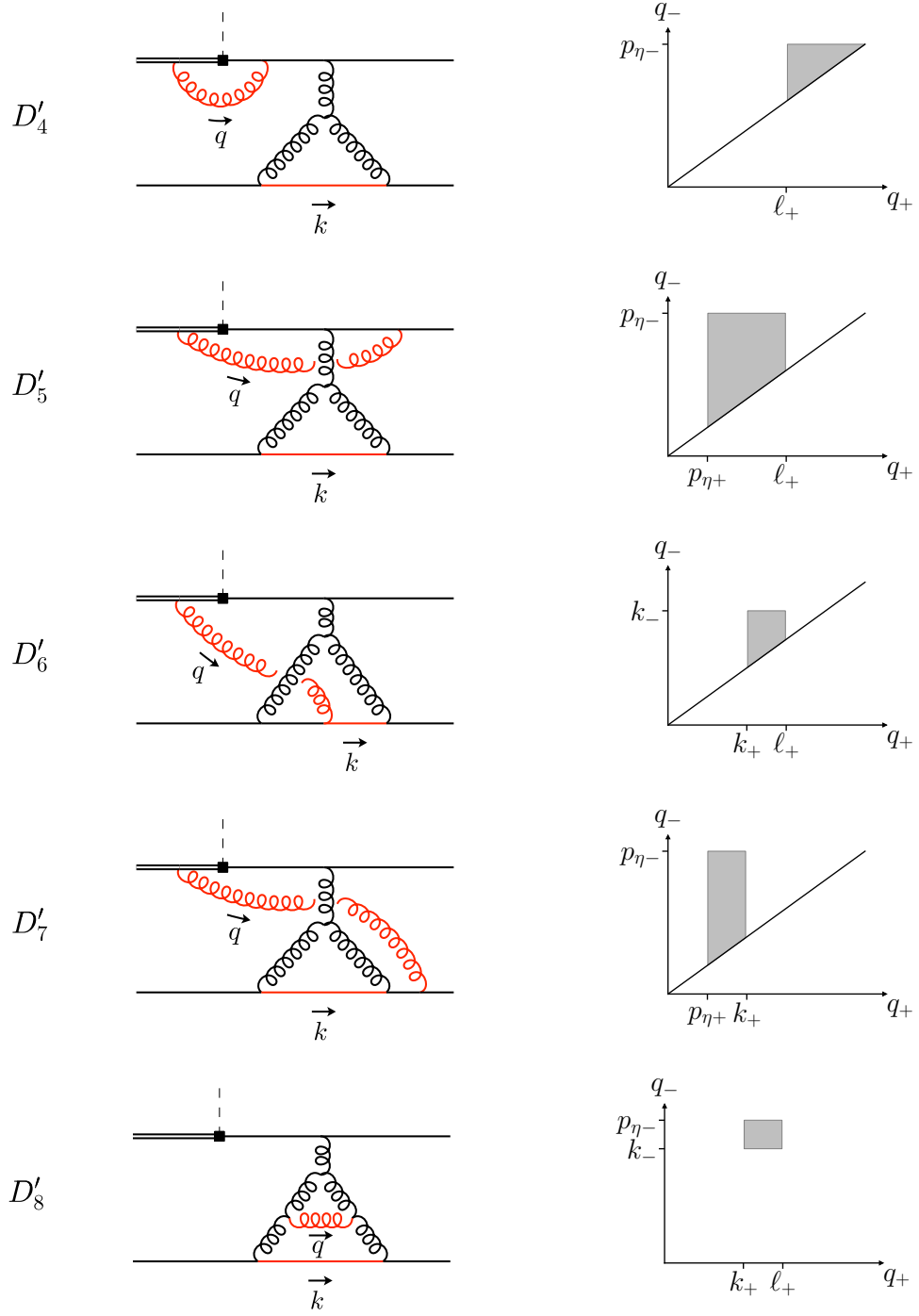


Figure 3.8: These diagrams generate the soft-gluon corrections to the non-Abelian soft-quark diagram D_2 . For each diagram we display the available phase space for the soft-gluon emission, where q^μ is the momenta of the soft gluon.

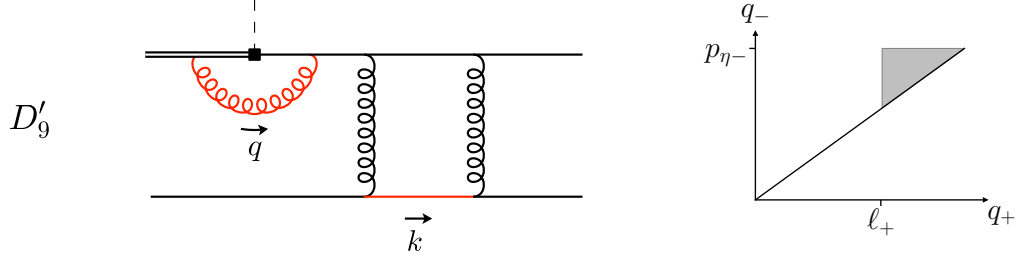


Figure 3.9: This diagram and the corresponding phase space generate the one-loop correction of a global Sudakov factor to the soft-quark box diagram D_1 .

Thus, the contribution of D'_4 is given by the same multiplicative factor as for T_2 (cf. (3.34)) with the Born exchanged by D_2

$$D'_4 \simeq D_2 \times \frac{\alpha_s}{4\pi} (-C_F) L^2. \quad (3.43)$$

The remaining diagrams are more involved because their phase space is sensitive to the soft-quark momentum k^μ . Collecting the contributions from all soft gluon corrections to D_2 that have the color structure $C_F - C_A/2$ related to gluon crossing gives

$$D'_5 + D'_6 + D'_7 \simeq D_2 \otimes \frac{\alpha_s}{4\pi} (-2) \left(C_F - \frac{C_A}{2} \right) \ln \frac{p_{\eta-}}{k_-} \ln \frac{\ell_+}{k_+}, \quad (3.44)$$

where $\otimes$ indicates a convolution in the sense that the logarithm becomes part of the integrand of the soft-quark integral. We note that diagrams D'_5 , D'_6 and D'_7 all generate different logarithms and only their sum gives the logarithms in (3.44). Diagram D'_8 generates the same logarithms as a single diagram, but with a purely non-Abelian color structure

$$D'_8 \simeq D_2 \otimes \frac{\alpha_s}{4\pi} (-C_A) \ln \frac{p_{\eta-}}{k_-} \ln \frac{\ell_+}{k_+}, \quad (3.45)$$

which cancels the C_A term in (3.44). Finally, the soft-gluon correction of the soft-quark loop gives

$$D'_5 + D'_6 + D'_7 + D'_8 \simeq D_2 \otimes \frac{\alpha_s}{2\pi} (-C_F) \ln \frac{p_{\eta-}}{k_-} \ln \frac{\ell_+}{k_+}. \quad (3.46)$$

We notice that we can identify the relevant phase space from only one diagram, which is the purely non-Abelian diagram D'_8 . It will turn out to be an important insight for general soft-gluon corrections that it is sufficient to consider the large- N_c limit, where only planar diagrams contribute, to determine the phase space of the double logarithm that eventually exponentiates.

The corrections to the box diagram D_1 are more involved for reasons that we will only be able to understand once we develop a more systematic understanding of the Dirac structure in the next section. We therefore postpone a detailed discussion to a later section and only note a few observations here. The additional complication is that, in contrast to the soft-gluon corrections to the non-Abelian diagram D_2 that factorized for every single diagram, corrections to the box diagram D_1 do not factorize at the level of single diagrams. Instead, in some cases only the sum of certain subsets of diagrams factorizes to the box. Before we start analyzing the diagrams, we note that in Figures with relevant corrections to D_1 there are one or multiple Feynman diagrams with the same phase-space constraint applying to all Feynman diagrams in a given Figure. Similar to the tree-level diagram T_2 and the non-Abelian soft-quark diagram D_2 , also the box diagram D_1 receives

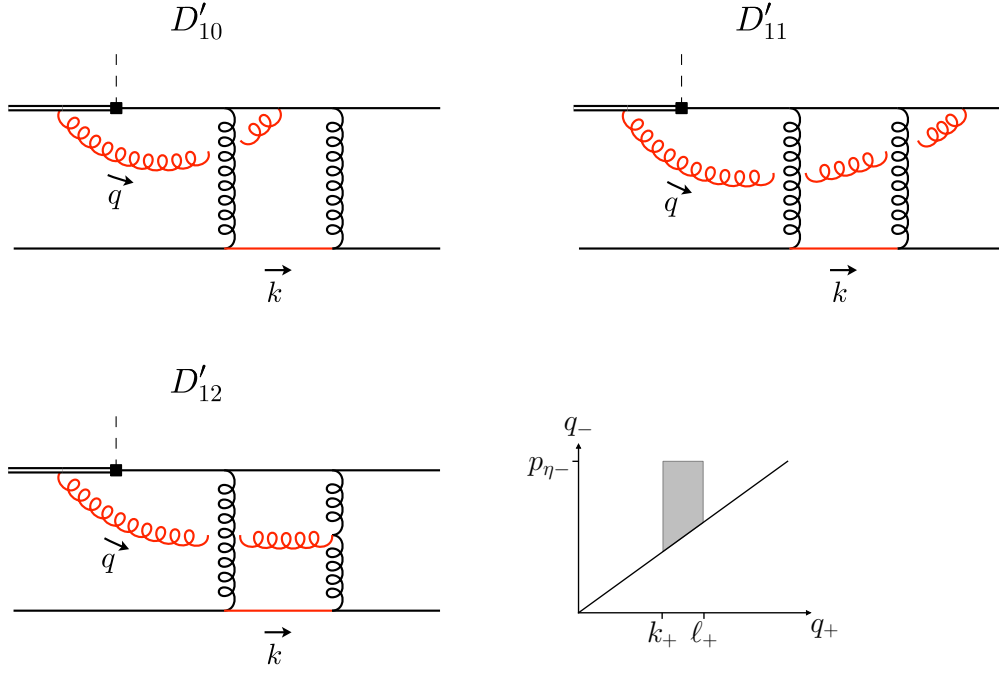


Figure 3.10: These diagrams are soft-gluon corrections to the box itself and all have the same phase space. We note that diagram D'_{11} also contributes in another phase space region, which is canceled by the diagram where the soft gluon attaches to the spectator instead of the active quark.

a soft gluon correction of the heavy-to-light vertex (see Fig. 3.9), which leads to a correction in the form of a global Sudakov logarithm

$$D'_9 \simeq D_1 \times \frac{\alpha_s}{4\pi} (-C_F) L^2. \quad (3.47)$$

Most of the remaining diagrams that contribute at the double-logarithmic level to the soft-gluon corrections of the box diagram D_1 do not factorize at the level of single diagrams. For instance, the diagrams shown in Fig. 3.10 give

$$\begin{aligned} D'_{10} &\simeq -\frac{\xi_0}{\bar{u}_0^3} \left(\frac{\alpha_s}{4\pi}\right)^2 \int_{\ell_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{\ell_+} \frac{dk_+}{k_+} \ln\left(\frac{p_{\eta-}^2}{k_+\ell_+}\right) \ln\left(\frac{\ell_+}{k_+}\right) C_F \left(C_F - \frac{C_A}{2}\right) (6 + 4\bar{u}_0), \\ D'_{11} &\simeq -\frac{\xi_0}{\bar{u}_0^3} \left(\frac{\alpha_s}{4\pi}\right)^2 \int_{\ell_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{\ell_+} \frac{dk_+}{k_+} \ln\left(\frac{p_{\eta-}^2}{k_+\ell_+}\right) \ln\left(\frac{\ell_+}{k_+}\right) \left(C_F - \frac{C_A}{2}\right)^2 (2\bar{u}_0), \\ D'_{12} &\simeq -\frac{\xi_0}{\bar{u}_0^3} \left(\frac{\alpha_s}{4\pi}\right)^2 \int_{\ell_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{\ell_+} \frac{dk_+}{k_+} \ln\left(\frac{p_{\eta-}^2}{k_+\ell_+}\right) \ln\left(\frac{\ell_+}{k_+}\right) \frac{C_A}{2} \left(C_F - \frac{C_A}{2}\right) (2\bar{u}_0), \end{aligned} \quad (3.48)$$

where none of the diagrams factorize to D_1 , which can be deduced from the different $\bar{u}_0$ structure

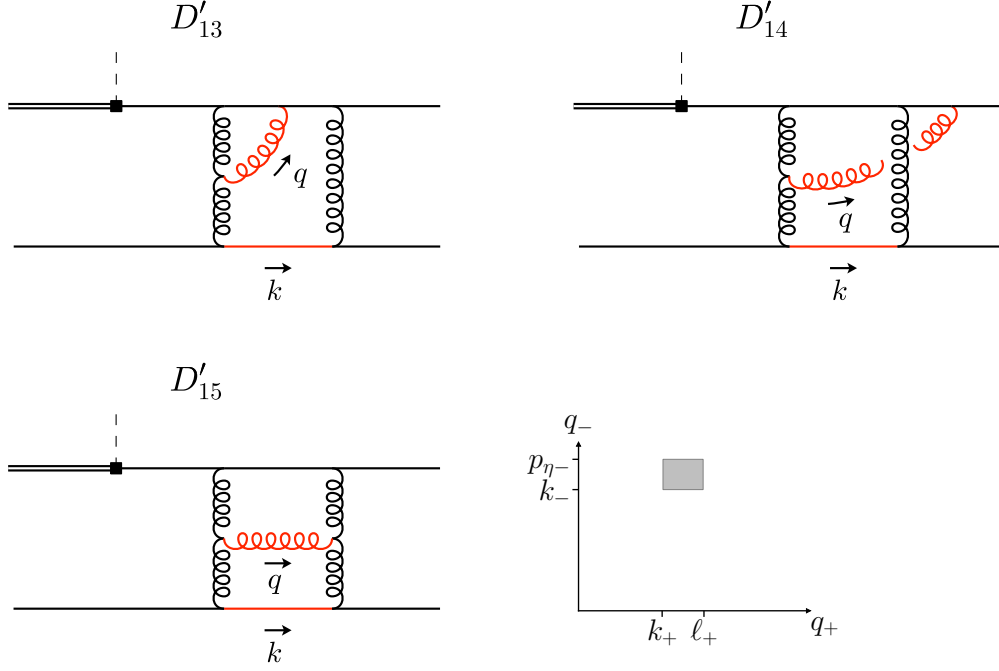


Figure 3.11: These diagrams are soft-gluon corrections to the box itself and all have the same phase space. We note that diagram D'_{14} also contributes in another phase space region, which is canceled by the diagram where the soft gluon attaches to the spectator instead of the active quark. On the other hand, diagram D'_{13} contributes only for the given phase space and its equivalent where the soft gluon attaches to the spectator quark vanishes because there is no double-logarithmic phase space.

(cf. (3.28)). However, the sum of all three diagrams factorizes to D_1

$$\begin{aligned}
 D'_{10} + D'_{11} + D'_{12} &\simeq -\frac{\xi_0}{\bar{u}_0^3} \left(\frac{\alpha_s}{4\pi} \right)^2 \int_{\ell_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{\ell_+} \frac{dk_+}{k_+} \ln \left(\frac{p_{\eta-}^2}{k_+ \ell_+} \right) \ln \left(\frac{\ell_+}{k_+} \right) C_F \left(C_F - \frac{C_A}{2} \right) (6 + 6\bar{u}_0) \\
 &= D_1 \otimes \frac{\alpha_s}{4\pi} \left(- \left(C_F - \frac{C_A}{2} \right) \right) \ln \left(\frac{p_{\eta-}^2}{k_+ \ell_+} \right) \ln \left(\frac{\ell_+}{k_+} \right).
 \end{aligned} \tag{3.49}$$

A similar pattern holds for the diagrams in Fig. 3.11, while the diagram in Fig. 3.12 factorizes directly to D_1 and we omit intermediate results. When considering all diagrams in Figs. 3.10, 3.11, and 3.12 we obtain

$$\sum_{i=10}^{16} D'_i \simeq D_1 \otimes \frac{\alpha_s}{2\pi} (-C_F) \ln \frac{p_{\eta-}}{k_-} \ln \frac{\ell_+}{k_+}, \tag{3.50}$$

which is the equivalent of (3.44) for the corrections to the box in D_1 rather than the loop in the non-Abelian diagram D_2 . We notice that in diagrams that do not directly factorize to D_1 there is always the soft-gluon momentum q^μ flowing through the active quark propagator of the box, which will be an important observation when we discuss why these diagrams do factorize at the level of single

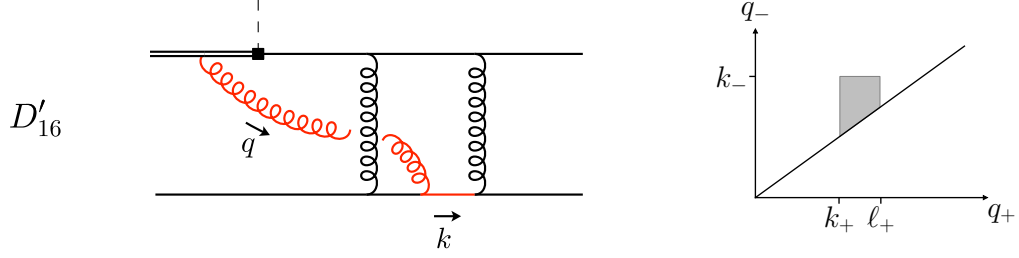


Figure 3.12: This diagram and the corresponding phase space contribute to the soft gluon corrections to the soft-quark box diagram D_1 .

diagrams. Adding all diagrams relevant for the interplay of one-loop soft gluons with the one-loop soft-quark diagrams D_1 and D_2 we obtain the total contribution

$$F_{\text{mixed}}^{(2)}(\gamma) \simeq \xi_0 L^4 \left(-3C_F^2 \frac{1+\bar{u}_0}{\bar{u}_0^3} + \frac{C_A C_F}{2\bar{u}_0^3} \right) + \xi_0 L^4 \left(-C_F^2 \frac{1+\bar{u}_0}{2\bar{u}_0^3} + \frac{C_A C_F}{12\bar{u}_0^3} \right), \quad (3.51)$$

where the first term stems from the global Sudakov factor (cf. (3.28) and (3.31)), and the second term reflects the non-trivial interplay of soft-gluon and soft-quark effects. Combining the results of the pure soft-quark diagrams (3.41) with the ones from pure soft gluon diagrams (3.42), and from mixed soft-gluon and soft-quark diagrams (3.51) gives the final result for the form factor $F(\gamma)$ at NNLO

$$F^{(2)}(\gamma) \simeq \xi_0 L^4 \left(-C_F^2 \frac{7+10\bar{u}_0}{6\bar{u}_0^3} + C_A C_F \frac{3-2\bar{u}_0}{12\bar{u}_0^3} \right), \quad (3.52)$$

which has a very non-trivial dependence on both $\bar{u}_0$ and color due to the different effects at play. We emphasize that we perform an independent cross-check of this result in the next chapter.

3.2 SOFT-QUARK DIAGRAMS TO ALL ORDERS

We now perform a more in-depth analysis of soft-quark diagrams with the aim to develop a systematic understanding of the double-logarithmic series to all orders. We continue to work in hybrid gauge, but since we do not consider soft-gluon corrections in this section all gluons are in LCG and, hence, there are no leading power attachments to the heavy quark. We define soft-quark diagrams as diagrams that have one on-shell spectator-quark propagator for every loop momentum. This implies that any gluon attaching to the spectator-quark line has momentum $k_i - k_{i+1}$, which is eikonalized if the loop momenta obey strong-rapidity ordering

$$\begin{aligned} \ell_- &\ll k_{n-} \ll k_{n-1,-} \ll \dots \ll k_{1-} \ll p_{2-}, \\ \ell_+ &\gg k_{n+} \gg k_{n-1,+} \gg \dots \gg k_{1+} \gg p_{2+}. \end{aligned} \quad (3.53)$$

The ordering criteria in (3.53) lead to nested integrals of the form

$$\begin{aligned} &\int_{\ell_-}^{p_{2-}} \frac{dk_{n-}}{k_{n-}} \int_{k_{n-}}^{p_{2-}} \frac{dk_{n-1,-}}{k_{n-1,-}} \dots \int_{k_{2-}}^{p_{2-}} \frac{dk_{1-}}{k_{1-}} \int_{m_2^2/k_{n-}}^{\ell_+} \frac{dk_{n+}}{k_{n+}} \int_{m_2^2/k_{n-1,-}}^{k_{n+}} \frac{dk_{n-1,+}}{k_{n-1,+}} \dots \int_{m_2^2/k_{1-}}^{k_{2+}} \frac{dk_{1+}}{k_{1+}} \\ &= \frac{1}{n!(n+1)!} L^{2n}. \end{aligned} \quad (3.54)$$

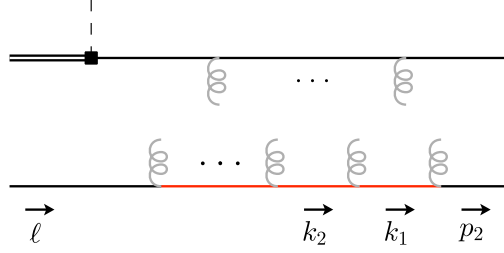


Figure 3.13: A general soft-quark diagram. The gluons (in gray) are left implicit.

A diagram illustrating the definition of soft-quark diagrams is shown in Fig. 3.13. In a soft-quark diagram any gluon that is emitted from the spectator-quark line must either attach to the active-quark line or to an interaction of the spectator-quark line and active-quark line formed by one or more other gluons. In any other case it is impossible to meet the condition that there is one on-shell spectator quark for every loop momentum. This implies that the class of soft-quark diagrams is relatively large. All of these diagrams are eikonalized by the strong-rapidity ordering in (3.53) and, therefore, have to be considered. It turns out that only a small subset of soft-quark diagrams contributes at the double-logarithmic level. Nevertheless, it is important to understand why any other soft-quark diagram cannot contribute to gain a reliable understanding of the all-order systematics.

Before we start with the calculation we make a number of observations that will simplify the analysis significantly. The first one concerns the power counting. Following the strategy laid out in Sec. 3.1.1, any double logarithm originates from an integral of the form

$$\int \frac{dk_{n+}}{k_{n+}} \int \frac{dk_{n-}}{k_{n-}}, \quad (3.55)$$

where the $k_{\perp}$ has already been integrated out by means of the on-shell condition (3.19). This implies that even if we assign a power counting to the loop momenta k_n according to the strong-rapidity ordering in (3.53), for any term that eventually contributes to the double-logarithmic series the power counting of the loop momenta cancels out. Hence, we can effectively treat powers of $k_{n\pm}$ separately from regular power counting. In practice this means that we can ignore the power counting of the loop momenta once the denominators are expanded according to (3.53). Keeping in mind that there are no attachments to the heavy-quark line in LCG, the power of m_b is fixed and the power of the only other dimensionful parameter m_{η} is fixed on dimensional grounds in the double-logarithmic approximation. Thus, the only parameter that carries a power counting is γ . Consequently, power counting in the double-logarithmic approximation effectively amounts to counting powers of γ . We now come to the second observation and use it as an example to further illustrate the power counting. By always assigning the loop momenta to the spectator quark, the overall momentum routing is fixed. This leads to two insights. First, it allows us to use building blocks with well-defined dependence on momenta. Second, any active-quark propagator² depends on the momentum $p_{\eta} - k_n$ or $p_{\eta} - \ell$. For instance, the power expansion of the former is given by

$$A \frac{i(\not{p}_{\eta} - \not{k}_n + m_1)}{(p_{\eta} - k_n)^2 - m_1^2} = -iA \frac{\not{p}_{\eta}}{2k_{n+}} + O(\gamma^{-1}), \quad (3.56)$$

² Once we consider diagrams with crossed gluons in the next section, there are also active-quark propagators where p_{η} is replaced by p_1 and more than one loop momentum. But the argument can be trivially extended to this case.

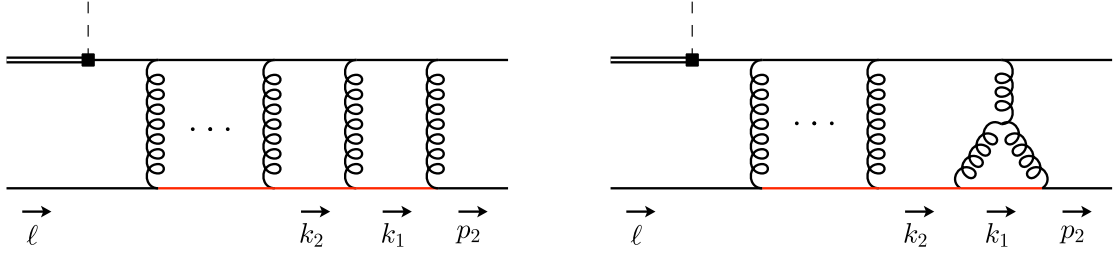


Figure 3.14: Sample diagrams that generate leading logarithmic corrections from soft-quark configurations in light-cone gauge.

where A is some test Dirac structure. Consequently, any Dirac structure A that appears directly to the left of such an active-quark propagator as in (3.56) and has a $\not{n}$ to its right receives a power suppression. We refer to this principle as power suppression due to $\not{n}$ to the right (PSNR) and it is very helpful to constrain the kind of Dirac structures that can contribute. It is instructive to understand the difference to regular SCET power counting adjusted to the strong-rapidity ordering in (3.53). In this case the propagator would be enhanced by the inverse power of the small momentum component k_{n+} and Dirac structures other than $\not{n}$ might nevertheless contribute because it is not clear whether there could be an enhancement at a later point in the iteration. In the hybrid power and double-logarithmic counting we apply here, there is exactly one term—the one on the r.h.s of (3.56)—which is $\mathcal{O}(1)$ and all other terms are power suppressed in γ .

3.2.1 Contributing soft-quark diagrams

In this section we calculate all soft-quark diagrams that contribute at the double-logarithmic level. This means that we concentrate on a subset of soft-quark diagrams and leave an analysis why this is justified to the next section. The subset that we focus on here is given by the equivalents of the soft-quark diagrams that contribute at NLO (cf. Fig. 3.6) and NNLO (cf. Fig. 3.7). To this end, we consider the ladder diagrams shown in Fig. 3.14, where the active- and spectator-quark lines are connected either by a gluon directly or by a three-gluon vertex, and the spectator-quark propagator becomes on-shell as indicated by the red line in the diagrams.

Our goal in the following is to determine the coefficients of the integrals in (3.54). To this end, we now use the insight that the momentum routing allows us to decompose the diagrams into iterated building blocks.

LADDER DIAGRAMS: We first consider Abelian diagrams without three-gluon vertices. Starting the series of building blocks at the η_c meson side, the first iteration is simply given by

$$\begin{array}{c}
 \overline{\text{---}} \xrightarrow{p_\eta - p_2} \text{---} \\
 | \\
 \text{---} \xrightarrow{k_1} \text{---} \xrightarrow{p_2} \text{---}
 \end{array}
 = it^a \gamma^\mu \mathcal{P}_\eta it^a \gamma^\nu G_{\mu\nu}(k_1 - p_2) \approx \frac{1}{k_{1+}} \sum_{i=1}^3 c_i^{(0)} \Gamma_i(k_1) \gamma_5, \quad (3.57)$$

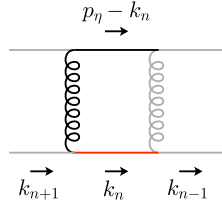
where $\mathcal{P}_\eta$ is the Dirac projector of the η_c meson (cf. (2.1)), $G_{\mu\nu}$ is the gluon propagator in LCG (cf. (3.3)) and we defined the Dirac structures³

$$\Gamma_i(k_1) = \left\{ \gamma \not{p}, 1, \frac{\not{k}_{1,\perp} + k_{1+} \not{n}}{\bar{u}_0 m_\eta} \right\}, \quad (3.58)$$

where $\Gamma_2(k_1)$ and $\Gamma_3(k_1)$ are power suppressed with respect to $\Gamma_1(k_1)$ by one power of γ . However, we already anticipated that $\Gamma_1(k_1)$ will receive a power suppression in the next iteration step because of PSNR. The diagram in (3.57) and similar diagrams below should be read as follows. Black (or red) lines stand for lines that are part of the building block and gray lines indicate that the building block has open spinor or Lorentz indices. Explicit calculation of the Dirac structure in (3.57) yields

$$c_1^{(0)} = c_2^{(0)} = -c_3^{(0)} = -\frac{\pi \alpha_s C_F f_\eta}{N_c \gamma \bar{u}_0}. \quad (3.59)$$

The fact that the $B_c \rightarrow \eta_c$ form factor at large hadronic recoil is a power-suppressed object in terms of SCET power counting also translates to the generalized power counting we apply here. Consequently, we will see that any term that eventually contributes to the form factor at leading power receives *precisely one* power suppression in γ . For instance, $\Gamma_1(k_1)$ can still receive one power suppression and we know it will because of PSNR. On the other hand, $\Gamma_{2/3}(k_1)$ already have one power suppression in γ and can therefore only be dressed with LP interactions. The strategy of the remaining calculation is to take the r.h.s of (3.57), generalize it to arbitrary loop momenta k_n and insert it as input for the ladder rung with loop momentum k_n . If this iteration is closed, i. e. if it does not generate new Dirac structures, then we derived iteration equations for the coefficients $c_i^{(n)}$. The denominator structure of the next iteration step is given by



$$\approx \int \frac{d^4 k_n}{(2\pi)^4} \frac{\text{Numerator}}{[k_n^2 - m_2^2] (-k_{n+} p_{\eta-}) (-k_{n+1,+} k_{n-}) k_{n+}}, \quad (3.60)$$

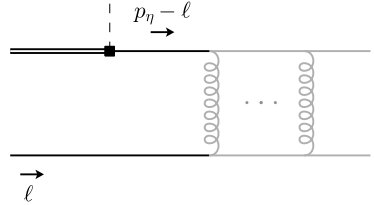
where the last factor, k_{n+} , is inherited from (3.57) with $k_1 \rightarrow k_n$, which is graphically represented by the right gray part of the diagram, and the numerator includes denominators specific to the LCG gluon propagator (3.3). From the denominator structure (3.60) we can deduce that terms in the numerator have to provide exactly one power of k_{n+} and no power of k_{n-} to contribute at the double-logarithmic level according to (3.54). Applying this constraint and dropping power-suppressed terms, the result can again be brought into the general form (3.57) leading to the iteration equation⁴

$$\left\{ \begin{array}{l} c_1^{(n)} \\ c_2^{(n)} \\ c_3^{(n)} \end{array} \right\} \simeq \frac{\alpha_s C_F}{2\pi} \int_{k_{n+1,-}}^{p_{2-}} \frac{dk_{n-}}{k_{n-}} \int_{m_2^2/k_{n-}}^{k_{n+1,+}} \frac{dk_{n+}}{k_{n+}} \left\{ \begin{array}{l} 0 \\ -(1 - \bar{u}_0) c_1^{(n-1)} + c_2^{(n-1)} - c_3^{(n-1)}, \\ -\bar{u}_0 c_1^{(n-1)} + c_3^{(n-1)}, \end{array} \right. \quad (3.61)$$

³ We note that the Dirac structure used in [1] has more terms than (3.58), but all additional terms do not contribute due to PSNR or two power suppressions.

⁴ Notice that the coefficients $c_i^{(n)}$ include factors of $\alpha_s/(2\pi)$, which is different from the notation we used in the previous section.

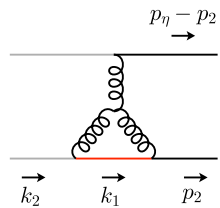
which is valid for $n \geq 1$. To be precise, the iteration generates e. g. terms of the form $\not{n}\not{n}$ that are already power suppressed in γ and, hence, can be dropped already at this iterations step because of PSNR. We emphasize that the iteration equation (3.61) does not produce any new Dirac structures and we therefore derived closed iteration equations for the coefficients $c_i^{(n)}$ for Abelian ladder rungs. Additionally, (3.61) confirms the prediction that $\Gamma_1(k_n)$ receives a power suppression since $c_1^{(n)} = 0$ and $c_1^{(n-1)}$ only appears in coefficients for Dirac structures $\Gamma_{2/3}(k_n)$ which have already one power suppression. To arrive at a final result for an n -loop ladder diagram (with $n + 1$ rungs) we have to feed the Dirac structures and their coefficients into the leftmost part of the diagram that does not depend on loop momenta, includes the heavy-to-light vertex, and closes the Dirac and color traces by means of the B_c meson projector $\mathcal{P}_B$. This step yields the final formula



$$\approx \frac{f_B m_B}{2\bar{u}_0^2 m_\eta^2} \left[(1 - \bar{u}_0) c_1^{(n)} - c_2^{(n)} + 2c_3^{(n)} \right]. \quad (3.62)$$

Given (3.62) in combination with (3.59) and (3.61) one straightforwardly verifies that the above prescription correctly reproduces the contributions from the tree-level, one-loop and two-loop ladder diagrams given in (3.2), (3.28) and (3.39), respectively.

NON-ABELIAN CONTRIBUTIONS: The non-Abelian diagrams with one three-gluon vertex are also eikonalized by the strong rapidity ordering (3.53). In this case already the first three-gluon rung is a one-loop contribution in contrast to the first Abelian rung which was tree-level. As we will see in the next section, we could, in principle, decompose this into a tree-level first rung with an outgoing gluon and a subsequent one-loop building block with an incoming gluon. At the level of this section this decomposition would be an unnecessary layer of complication since these two building blocks only ever contribute in combination with each other. A similar line of reasoning also applies for the non-Abelian equivalent of (3.60) and we therefore treat the non-Abelian building blocks with one additional loop with respect to their Abelian counterparts. The non-Abelian equivalent of (3.57) has the denominator structure



$$\approx \int \frac{d^4 k_1}{(2\pi)^4} \frac{\text{Numerator}}{[k_1^2 - m_2^2] (-k_{1+} p_{2-}) (-k_{2+} p_{2-}) (-k_{2+} k_{1-})}, \quad (3.63)$$

and hence the double-logarithmic part of the numerator must be independent of k_{1+} and k_{1-} . Explicit calculation gives $\hat{c}_1^{(1)} = \hat{c}_3^{(1)} = 0$ and

$$\hat{c}_2^{(1)} \simeq \frac{\pi \alpha_s C_F f_\eta}{N_c \gamma \bar{u}_0} \frac{\alpha_s C_A}{4\pi} \int_{k_{2-}}^{p_{2-}} \frac{dk_{1-}}{k_{1-}} \int_{m_2^2/k_{1-}}^{k_{2+}} \frac{dk_{1+}}{k_{1+}}. \quad (3.64)$$

Again, the result can be given in terms of the three Dirac structures introduced in (3.58). We denote the results of non-Abelian iteration equations with a hat to avoid clash of notation with the Abelian iteration equations. But it is understood that the result of a non-Abelian iteration equation can

enter as input into an Abelian iteration equation and vice versa. The denominator structure of an insertion of a three-gluon vertex at the n -th loop is given by

$$\approx \int \frac{d^4 k_{n-1}}{(2\pi)^4} \frac{d^4 k_n}{(2\pi)^4} \quad (3.65)$$

$$\times \frac{\text{Numerator}}{[k_{n-1}^2 - m_2^2] [k_n^2 - m_2^2] (-k_{n-1,+} p_{\eta-}) (-k_{n+} k_{n-1,-}) (-k_{n+1,+} k_{n-1,-}) (-k_{n+1,+} k_{n-}) k_{n-1,+}}.$$

Hence, terms in the numerator need to have a factor $k_{n-1,+} k_{n-1,-}$ and should be independent of the loop momentum k_n to contribute at the double-logarithmic level. Collecting these terms gives

$$\left. \begin{matrix} \hat{c}_1^{(n)} \\ \hat{c}_2^{(n)} \\ \hat{c}_3^{(n)} \end{matrix} \right\} \simeq \left(\frac{\alpha_s}{4\pi} \right)^2 C_A C_F \int_{k_{n+1,-}}^{p_{n-}} \frac{dk_{n-}}{k_{n-}} \int_{m_2^2/k_{n-}}^{k_{n+1,+}} \frac{dk_{n+}}{k_{n+}} \int_{k_{n-}}^{p_{2-}} \frac{dk_{n-1,-}}{k_{n-1,-}} \int_{m_2^2/k_{n-1,-}}^{k_{n+}} \frac{dk_{n-1,+}}{k_{n-1,+}} \left\{ \begin{matrix} 0, \\ -2\bar{u}_0 c_1^{(n-2)} + 2c_3^{(n-2)}, \\ 0, \end{matrix} \right. \quad (3.66)$$

which is valid for $n \geq 2$. Again one easily verifies that this procedure correctly reproduces the contributions from the non-Abelian one-loop and two-loop diagrams in (3.31) and (3.40), respectively.

Inspecting the iteration equations for the Abelian and non-Abelian rungs leads to an interesting insight. Any non-Abelian building block generates exclusively c_2 -type⁵ coefficients, but can only produce these terms given c_1 - or c_3 -type coefficients as input. At the same time, in the absence of c_1 - and c_3 -type coefficients, the Abelian iteration equations (3.61) mix c_2 -type coefficients only into c_2 -type coefficients without any other contribution. This implies that any diagram can have at most one three-gluon vertex and all other rungs must be of the Abelian type. This is a new insight compared to the fixed-order analysis in the previous section since a diagram with two non-Abelian building blocks only arises first at the three-loop level.

At this point we obtained iteration equations that are predictive, but we can assemble them into a more convenient form. The first step in this direction is to decouple the evolution of the coefficient c_1 from those of the c_2 and c_3 coefficients. This is achieved by performing the change of variables

$$c_2'^{(n)} = c_2^{(n)} - c_1^{(n)}, \quad c_3'^{(n)} = c_3^{(n)} - \bar{u}_0 c_1^{(n)}. \quad (3.67)$$

After this change of variables the primed coefficients obey the same iteration equations (3.61) and (3.66), but without the $c_1^{(n)}$ coefficients. In these variables the tree-level values are given by

$$c_2'^{(0)} = 0, \quad c_3'^{(0)} = \frac{\pi \alpha_s C_F f_\eta}{N_c \gamma \bar{u}_0} (1 + \bar{u}_0). \quad (3.68)$$

In the next step we combine the one-step Abelian iteration equation (3.61) with the two-step non-Abelian iteration equation (3.66). To this end, it is important to realize that the non-Abelian iteration equation can only mix at most one time with the Abelian iteration equation and that this single

5 We refer to coefficients as c_i -type to acknowledge that there are both hatted and unhatted coefficients.

mixing can be split into two parts. The first one can be done by a regular Abelian $c_3^{(n-2)}$ to $c_3^{(n-1)}$ iteration with color factor C_F and the second one can be done by an additional effective $c_3^{(n-1)}$ to $c_2^{(n)}$ iteration with C_A color factor. The iteration equation formed in this way is only valid for $n \geq 2$, but it can be adjusted to be valid for $n \geq 1$ if we subtract the difference. This procedure leads to the iteration equation

$$\left. \begin{matrix} c_2'^{(n)} \\ c_3'^{(n)} \end{matrix} \right\} \simeq \frac{\alpha_s}{4\pi} \int_{k_{n+1,-}}^{p_{2-}} \frac{dk_{n-}}{k_{n-}} \int_{m_2^2/k_{n-}}^{k_{n+1,+}} \frac{dk_{n+}}{k_{n+}} \left\{ \begin{matrix} 2C_F c_2'^{(n-1)} + (C_A - 2C_F) c_3'^{(n-1)} - \delta_{n1} C_A c_3'^{(0)}, \\ 2C_F c_3'^{(n-1)}. \end{matrix} \right. \quad (3.69)$$

To be precise, (3.69) correctly iterates the tree-level coefficients $c_i^{(0)}$ or higher coefficients $c_i^{(n)}$ to the next loop level for both Abelian and non-Abelian effects, but it does not account for terms that enter only at one-loop through diagrams with the three-gluon vertex in the rightmost rung (3.64). It would be straightforward to also include these diagram into (3.69), but this would lead to a mixing with $\bar{u}_0$ -dependence and it is more convenient to treat this set of diagrams in a separate contribution to the final result. We now define auxiliary functions that allow us to express the iteration equations of the coefficients in terms of integral equations for these auxiliary functions. Eventually this representation will be useful when we derive a compact expression for the order-by-order solution. The first auxiliary function is recursively defined by

$$f_1(q_+, q_-) = 1 + \frac{\alpha_s}{4\pi} \int_{q_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{q_+} \frac{dk_+}{k_+} \left[2C_F f_1(k_+, k_-) \right] \quad (3.70)$$

and its on-shell version with $q_+ = \ell_+ = m_2$ and $q_- = \ell_- = m_2$ is related to the $c_3'^{(n)}$ coefficients by

$$\sum_{n=0}^{\infty} c_3'^{(n)} = \frac{\pi \alpha_s C_F f_\eta}{N_c \gamma \bar{u}_0} (1 + \bar{u}_0) f_1(m_2, m_2). \quad (3.71)$$

Similarly, we recursively define the second auxiliary function as

$$f_2(q_+, q_-) = 1 + \frac{\alpha_s}{4\pi} \int_{q_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{q_+} \frac{dk_+}{k_+} \left[2C_F f_2(k_+, k_-) + \left(C_F - \frac{C_A}{2} \right) f_1(k_+, k_-) + \frac{C_A}{2} \right]. \quad (3.72)$$

Its on-shell version is related to the coefficients making up the final trace (cf. (3.62)) by

$$\sum_{n=0}^{\infty} (2c_3'^{(n)} - c_2'^{(n)}) = \frac{\pi \alpha_s C_F f_\eta}{N_c \gamma \bar{u}_0} 2(1 + \bar{u}_0) f_2(m_2, m_2). \quad (3.73)$$

To understand the relation of $f_2(q_+, q_-)$ to the sum in (3.73) it is instructive to rewrite (3.69) as

$$\left. \begin{matrix} 2c_3'^{(n)} - c_2'^{(n)} \\ c_3'^{(n)} \end{matrix} \right\} \simeq \frac{\alpha_s}{4\pi} \int_{k_{n+1,-}}^{p_{2-}} \frac{dk_{n-}}{k_{n-}} \int_{m_2^2/k_{n-}}^{k_{n+1,+}} \frac{dk_{n+}}{k_{n+}} \left\{ \begin{matrix} 2C_F (2c_3'^{(n-1)} - c_2'^{(n-1)}) - (C_A - 2C_F) c_3'^{(n-1)} + \delta_{n1} C_A c_3'^{(0)}, \\ 2C_F c_3'^{(n-1)}. \end{matrix} \right. \quad (3.74)$$

Finally, we have to recover the contributions from diagrams with a three-gluon vertex in the rung next to the η_c meson. This can be done trivially by realizing that in the absence of $c_3^{(n)}$ coefficients,

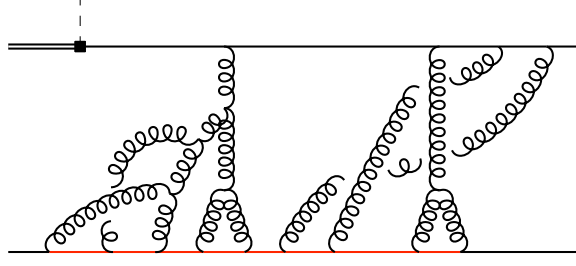


Figure 3.15: A soft-quark diagram at eight-loop order.

which is the case after a three-gluon vertex insertion, the $c_2^{(n)}$ coefficients obey the same iteration equations as the $c_3^{(n)}$ coefficients. Hence, we can use f_1 to account for them to all orders. Considering all prefactors and subtracting the tree-level as appropriate, the contribution of these diagrams to the form factor $F(\gamma)$ is given by

$$\frac{C_A \xi_0}{2C_F \bar{u}_0^3} (1 - f_1(m_2, m_2)). \quad (3.75)$$

Collecting all pieces the total contribution to the soft-overlap form factor $F(\gamma)$ by pure soft-quark diagrams is

$$F_{\text{soft quarks}}(\gamma) \simeq \xi_0 \left(2 \frac{1 + \bar{u}_0}{\bar{u}_0^3} f_2(m_2, m_2) + \frac{C_A}{2C_F \bar{u}_0^3} (1 - f_1(m_2, m_2)) - \frac{1}{\bar{u}_0^2} \right). \quad (3.76)$$

The different contributions to this equation can be understood as follows. The first term proportional to f_2 captures all Abelian soft-quark diagrams beyond tree-level and all non-Abelian diagrams except those with three-gluon vertex in the first rung next to the η_c meson. These kind of non-Abelian diagrams only start to contribute at NNLO, which is consistent with the fact that the C_A term in f_2 vanishes at NLO. This term depends on $\bar{u}_0$ through $(1 + \bar{u}_0)/\bar{u}_0^3$, which matches with our results at fixed order (cf. (3.28), (3.39) and (3.40)). The term in the middle describes all diagrams with a three-gluon vertex in the first rung next to the η_c meson and the $\bar{u}_0$ structure again matches with the fixed order results (cf. (3.31) and (3.40)).

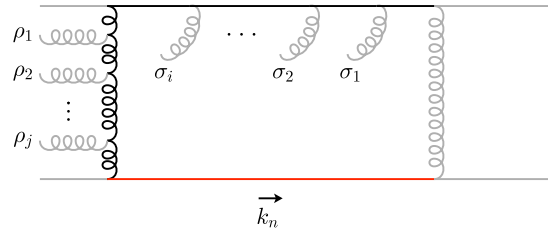
The implicit integral equations (3.70) and (3.72) can be solved in closed form in terms of modified Bessel functions [3, 78, 129]. Defining $z = \frac{\alpha_s C_F}{2\pi} L^2$, we find for the two functions that enter the soft-overlap form factor in (3.76),

$$\begin{aligned} f_1(m_2, m_2) &\stackrel{\text{soft quarks}}{=} \frac{I_1(2\sqrt{z})}{\sqrt{z}}, \\ f_2(m_2, m_2) &\stackrel{\text{soft quarks}}{=} \frac{C_F + C_A}{2C_F} \frac{I_1(2\sqrt{z})}{\sqrt{z}} + \frac{2C_F - C_A}{4C_F} I_0(2\sqrt{z}) - \frac{C_A}{4C_F}. \end{aligned} \quad (3.77)$$

3.2.2 Remaining soft-quark diagrams

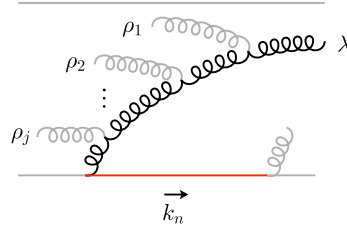
In the previous section we calculated the contributions from soft-quark diagrams without any crossings of gluons and at most one three-gluon vertex per rung in the ladder (cf. Fig. 3.14). We justified the simplification of focusing on this particular subset by anticipating that all remaining soft-quark diagram do not contribute in the double-logarithmic approximation. Keeping in mind that we aim to understand the double-logarithmic series of the soft-overlap term to all orders, this is a significant assumption. We remind the reader that we define soft-quark diagrams as configurations with

one on-shell spectator-quark propagator for each loop. These kinds of diagrams are eikonalized by the strong rapidity ordering in (3.53). But this set of diagrams is much larger than the subset we considered in the previous section. For instance, diagrams as complicated as the one shown in Fig. 3.15 fall into this category. It is *a priori* not clear why they do not contribute according to the method we laid out in Sec. 3.1.1, since any eikonalized diagram might contribute. By the end of this section we will be able to make a definite statement that any soft-quark diagram with strong-rapidity ordering (3.53), except the ones we considered in the previous section, does not contribute at the double-logarithmic level. In analogy to the analysis of the contributing diagrams we decompose any diagram into a set of building blocks. For complicated diagrams it is more convenient to work with one-loop building blocks which have incoming and outgoing gluons rather than constructing separate building blocks for every new topology as we did for the two-loop building block with a three-gluon vertex in the previous section (cf. e. g. (3.65)). It turns out that it is sufficient to introduce only two kinds of one-loop building blocks. The first one is given by



$$\equiv H_n^{\rho_j \dots \rho_2 \rho_1; \sigma_i \dots \sigma_2 \sigma_1}, \quad (3.78)$$

where all external gluons carry exclusively loop momenta k_{w_i} with $w_i > n$. The second one-loop building block is



$$\equiv L_n^{\rho_j \dots \rho_2 \rho_1 \lambda}, \quad (3.79)$$

where all outgoing gluons, i. e. those with Lorentz indices $\rho_1 \dots \rho_j$, carry exclusively loop momenta k_{w_i} with $w_i > n$. We note that these building blocks do not account for four-gluon vertices. We will see from the analysis below that there is need to consider them. For instance, using the building blocks given in (3.78) and (3.79) the two-loop soft-quark diagrams in Fig. 3.7 read

$$D'_1 \rightarrow H_2^i H_1^i H_0^i, \quad D'_2 \rightarrow H_2^i L_{1\lambda} H_0^{\lambda i}, \quad D'_3 \rightarrow L_{2\lambda} H_1^{\lambda i} H_0^i. \quad (3.80)$$

Another more involved example is the diagram in Fig. 3.15, which gives

$$L_{8\rho_5} L_{7\rho_4} L_{6\rho_5\rho_4}^{\rho_5\rho_4} L_{5\rho_3} H_4^{\rho_3\rho_2 i} L_{3\sigma_1} L_{2\sigma_2} L_{1\rho_1} H_0^{\rho_1; \sigma_2 \sigma_1}. \quad (3.81)$$

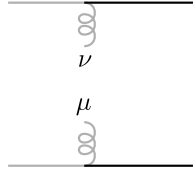
In analogy to the previous section we have to perform separate calculations for H_0 and H_n with $n > 0$ since the case $n = 0$ is fundamentally different due to the fact that the momentum p_2 flows through the gluon propagators of H_0 . For L -type building blocks, on the other hand, this is not required, because there cannot be a L_0 since we treat the η_c meson as a two-particle Fock state in the NR approximation. Before we start with the calculation we note that in this section we drop factors that are not relevant for double-logarithmic or power counting, e. g. $\bar{u}_0$, m_η , color factors or couplings, at our convenience. This is sufficient since none of the terms considered in this section will eventually give a new contribution with respect to the result of the previous section.

An important ingredient to our analysis is the gluon propagator in LCG, $\bar{n} \cdot A = 0$, given by

$$G^{\mu\nu}(k_n - k_w) \approx \frac{i}{k_{w+}k_{n-}} \left(g_{\perp}^{\mu\nu} - \frac{k_{n,w,\perp}^\mu \bar{n}^\nu + k_{n,w,\perp}^\nu \bar{n}^\mu}{k_{n-}} + \frac{k_{w+}}{k_{n-}} \bar{n}^\mu \bar{n}^\nu \right), \quad (3.82)$$

where $k_{n,w,\perp} \equiv k_{n\perp} - k_{w\perp}$, $w \geq n$, and we expanded the definition (3.3) in the strong-rapidity ordering (3.53). There are several comments in order that are important for the remaining steps of the calculation: First, according to the definition of LCG any contraction of the kind $G^{\mu\nu} \bar{n}_\mu$ vanishes. Hence, contractions with gluon propagators cannot generate any power of k_{n+} in the numerator. Second, for any given building block H_n or L_n the "-"-component that dominates according to the strong-rapidity ordering (3.53) is always that of loop momentum k_n , i. e. k_{n-} . This is, in particular, important to determine the dominant momentum components in denominators. Third, only the $g_{\perp}^{\mu\nu}$ -term of the LCG gluon propagator comes with only one inverse power of k_{n-} .

We will once again start from the η_c -meson side and focus first on H -type building blocks without gluon emissions from the active-quark line. Considering increasing numbers of gluon emissions from the gluon line the Lorentz structure becomes complicated. Before we combine it with the Dirac structure, it is beneficial to simplify the Dirac structure and drop terms that cannot contribute in the hybrid power and double-logarithmic counting. For H_0 this amounts to

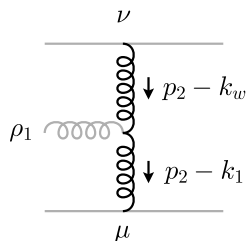


$$= it^a \gamma_\mu \mathcal{P}_\eta i t^b \gamma_\nu \propto \gamma_\mu \gamma_\nu \gamma_5 - \gamma \gamma_\mu \not{n} \gamma_\nu \gamma_5 \equiv \tilde{H}_{0,0,\mu\nu} + \tilde{H}_{0,1,\mu\nu}, \quad (3.83)$$

where $\propto$ indicates that we drop both proportionality factors and terms that are twice power suppressed in γ . Applying a single gluon propagator

$$G^{\mu\nu}(p_2 - k_1) = \frac{i}{k_{1+}p_{2-}} \left(g_{\perp}^{\mu\nu} + \frac{k_{1\perp}^\mu \bar{n}^\nu + k_{1\perp}^\nu \bar{n}^\mu}{p_{2-}} + \frac{k_{1+}}{p_{2-}} \bar{n}^\mu \bar{n}^\nu \right), \quad (3.84)$$

to the structures in (3.83) corresponds to an H_0 building block with no additional gluon emission. For $\tilde{H}_{0,0,\mu\nu}$, which is already power suppressed compared to $\tilde{H}_{0,1,\mu\nu}$, only the $g_{\perp}^{\mu\nu}$ in the gluon propagator can contribute, since all other terms lead to even more power suppression due to an additional inverse power of p_{2-} . This term generates a $\Gamma_2(k_1)$ Dirac structure (cf. (3.57)). We obtain the contribution from $\tilde{H}_{0,0,\mu\nu}$ to a single emission H_0 building block by combining it with two gluon propagators and a three-gluon vertex



$$= \frac{i}{k_{1+}p_{2-}} \left(g_{\perp}^{\mu\lambda_1} + \frac{k_{1\perp}^\mu \bar{n}^{\lambda_1} + k_{1\perp}^{\lambda_1} \bar{n}^\mu}{p_{2-}} + \frac{k_{1+}}{p_{2-}} \bar{n}^\mu \bar{n}^{\lambda_1} \right) \left(g_{\lambda_1\lambda_2} (2p_2 + \dots)^{\rho_1} \right. \\ \left. + g_{\lambda_1}^{\rho_1} (-p_2 + \dots)_{\lambda_2} + g_{\lambda_2}^{\rho_1} (-p_2 + \dots)_{\lambda_1} \right) \frac{i}{k_{w+}p_{2-}} \left(g_{\perp}^{\lambda_2\nu} + \frac{k_{w\perp}^{\lambda_2} \bar{n}^\nu + k_{w\perp}^\nu \bar{n}^{\lambda_2}}{p_{2-}} + \frac{k_{w+}}{p_{2-}} \bar{n}^{\lambda_2} \bar{n}^\nu \right), \quad (3.85)$$

where we omitted the color structure and k_w acts as a place holder for any loop momentum. For instance, from the topology of any possible diagram, it is clear that the "+"-component is dominated in the upper gluon propagator by a loop momentum k_w with $w > 1$. On the other hand, the $\perp$ -components might contain $k_{1\perp}$. Since there are two global denominators of gluon propagators with an inverse power of p_{2-} the term exhibits one additional power suppression in γ compared to the $g_{\perp}^{\mu\nu}$ -term in (3.84). In order to contribute, a term must cancel this power suppression in the numerator. In the gluon propagators any term—except the $g_{\perp}$ -terms—leads to even more power suppression that has to be canceled in the numerator, which is not possible. The first term in the three-gluon vertex is the only one that can generate a p_{2-} upon contraction with the $g_{\perp}$ -terms. The result is then a term that is once power suppressed and has a n^{ρ_1} Lorentz structure. It is straightforward to generalize this observation to arbitrary numbers of gluon emissions. In any additional gluon propagator only the $g_{\perp}$ -term can contribute to avoid two power suppressions, since there is only one additional source to cancel a power suppression, which is given by the equivalent term in an additional three-gluon vertex. The result is then once power suppressed and has the Lorentz structure $n^{\rho_1} \dots n^{\rho_m}$ for m gluon emissions. We can also conclude that four-gluon vertices cannot contribute, because it is impossible in this case to cancel a power suppression due to the missing momentum dependence. We will see that arguments along the same lines can also be used for the remaining contributions.

Turning to the contribution from $\tilde{H}_{0,1,\mu\nu}$, the discussion becomes more involved, since the Dirac structure itself is not yet power suppressed. Combining this term with the propagator in (3.84), we obtain terms proportional to

$$\gamma \frac{\not{k}\gamma_5}{k_{1+}p_{2-}}, \quad \gamma \frac{\not{k}_{1\perp}\gamma_5}{k_{1+}p_{2-}^2}, \quad \gamma \frac{\not{k}\gamma_5}{p_{2-}^2}, \quad (3.86)$$

which originate from the $g_{\perp}^{\mu\nu}$, $k_{1\perp}^{\mu} \bar{n}^{\nu}$, and $\bar{n}^{\mu} \bar{n}^{\nu}$ term in (3.84), respectively. The first term corresponds to a $\Gamma_1(k_1)$ Dirac structure and the remaining two terms to $\Gamma_3(k_1)$ (cf. (3.57)). Similarly, for single-gluon emission there is more than one possible combination of terms and we will discuss them one-by-one in the following. Considering the same terms for single-gluon emission as for $\tilde{H}_{0,0,\mu\nu}$ yields

$$\propto \gamma \frac{\not{k}\gamma_5}{p_{2-}} n^{\rho_1} \quad (3.87)$$

and for j gluon emissions

$$\propto \gamma \frac{\not{k}\gamma_5}{p_{2-}} n^{\rho_1} \dots n^{\rho_j}. \quad (3.88)$$

While the terms in (3.87) and (3.88) do not appear to be power suppressed, they will receive a power correction due to PSNR. If we instead consider the combination $g_{\perp}^{\mu\lambda_1}$, the second term in the three-gluon vertex, and the $\bar{n}^{\lambda_2} \bar{n}^{\nu}$ term in (3.85), the result for a single gluon emission is

$$\propto \gamma \frac{\gamma_{\perp}^{\rho_1}}{p_{2-}^2}, \quad (3.89)$$

which is once power suppressed in γ . For j emissions we take the equivalent terms in the first propagator and three-gluon vertex, where here and in the remaining discussion we refer to the first emission as the emission closest to the active-quark line in any H -type building block. Taking

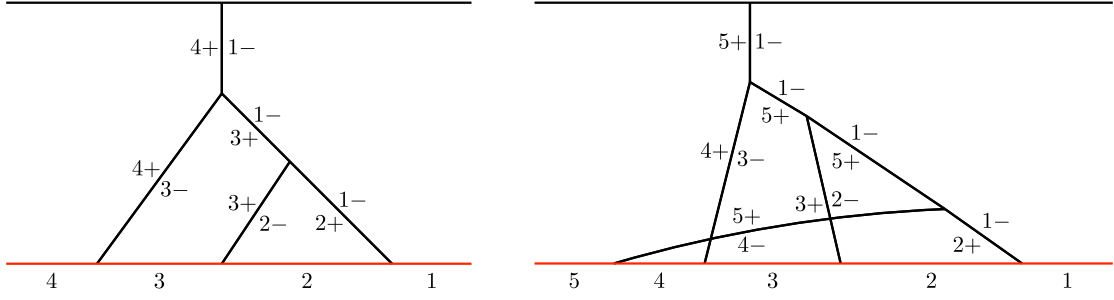


Figure 3.16: Parts of soft-quark diagrams with an H -type building block followed by L -type building blocks. The upper and lower line are the active- and spectator-quark line, while the lines that connect them represent gluons. The labels refer to the respective loop momenta and for simplicity we write only the index n of a loop momentum k_n . Any gluon is labeled by the loop momentum that dominates in a given light-cone direction according to the strong-rapidity ordering (3.53). In both diagrams k_2 is the loop momentum of the first L -type building block after the H -type building block. Importantly, only one gluon propagator is dominated by k_{2+} ($2+$ in the diagrams).

the $g_\perp$ -term in the remaining gluon propagators and the first term in the remaining three-gluon vertices, we obtain

$$\propto \gamma \frac{\gamma_\perp^{\rho_1}}{p_{2-}^2} n^{\rho_2} \dots n^{\rho_j}, \quad (3.90)$$

which is once power suppressed in γ as well.

The combination of $g_\perp^{\mu\lambda_1}$, the first term in the three-gluon vertex, and $k_{w\perp}^{\lambda_2} \bar{n}^\nu$ in (3.85) yields

$$\propto \gamma \frac{k_{w\perp} n^{\rho_1}}{p_{2-}^2}, \quad (3.91)$$

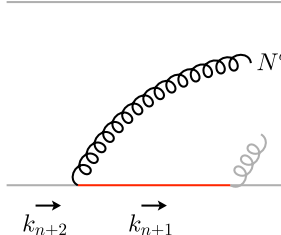
which is also once power suppressed in γ . We extrapolate this and the remaining contributions, that are already power suppressed, to j emissions via the same procedure as for the previous combination of terms, since any other choice leads to further power suppression. The result is

$$\propto \gamma \frac{k_{w\perp} n^{\rho_1}}{p_{2-}^2} \dots n^{\rho_j}, \quad (3.92)$$

which is power suppressed as are the other j gluon emission results. All other combinations of term do not contribute for $\tilde{H}_{0,1,\mu\nu}$. To understand this it is important to remember the form of the Dirac structure $\tilde{H}_{0,1,\mu\nu}$: If ν is in the $\perp$ -direction PSNR leads to a power suppression. For instance, if we consider the term $k_{w\perp}^\nu \bar{n}^{\lambda_2}$ in (3.85), there are two power suppressions, one due to an inverse p_{2-} and one due to PSNR.

Having computed all possible outputs of the first H -type building block with arbitrary numbers of emissions from the gluon line, we now analyze the impact of these terms as input for L -type building blocks to finalize the analysis of the first class of diagrams.

As illustrated in Fig. 3.16 an L -type building block with loop momentum k_{n+1} always inherits exactly one inverse power of $k_{n+1,+}$ from an H -type building block with loop momentum k_n . Hence, any L -type building block with loop momentum k_{n+1} and no additional gluon emission can be written as



$$= \frac{i}{k_{n+2,+} k_{n+1,-}} \left(g_{\perp\sigma}^\mu - \frac{k_{n+1,n+2,\perp}^\mu \bar{n}_\sigma + k_{n+1,n+2,\perp,\sigma} \bar{n}^\mu}{k_{n+1,-}} + \frac{k_{n+2,+}^\mu \bar{n}_\sigma}{k_{n+1,-}} \right) i\gamma_\mu i(-\not{k}_{n+1} + m_\eta \bar{u}_0) \frac{N^\sigma}{k_{n+1,+}}, \quad (3.93)$$

where we again omitted the color structure, $w > n + 1$ in this case, and

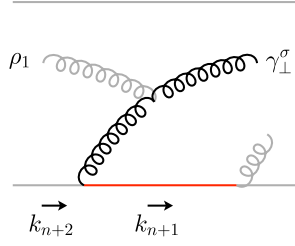
$$\frac{N^\sigma}{k_{n+1,+}}, \quad (3.94)$$

represents a contribution from an H -type building block. N^σ can thus be either of the type

$$\gamma \not{n}^\sigma, \quad \gamma_\perp^\sigma, \quad \gamma_\perp^\lambda n^\sigma, \quad \not{k}_{w,\perp} n^\sigma, \quad \not{k}_{n+1,\perp} n^\sigma. \quad (3.95)$$

All except the first term in (3.95) are already once power suppressed in γ . We will see below that all possible terms generated by emissions from L -type building blocks and H -type building blocks that are not H_0 also fall into these categories. We start by analyzing the effect of $N^\sigma = \gamma \not{n}^\sigma$ in (3.95). In this case the gluon propagator in (3.93) comes with two inverse powers of $k_{n+1,-}$. One of them has to be canceled in the numerator according to (3.55) in order to obtain a double-logarithmic contribution. The available options are via a factor of $k_{n+1,\perp}^2$ or directly via $k_{n+1,-}$. By means of the on-shell condition (3.19), $k_{n+1,\perp}^2$ always gives terms with either both $k_{n+1,-}$ and $k_{n+1,+}$ or none of the two. But in the case at hand $k_{n+1,+}$ already has the correct power for a double-logarithmic contribution while $k_{n+1,-}$ has one inverse power more than required. The only viable option is therefore $k_{n+1,-}$ from the spectator-quark propagator. This term, however, comes with a $\not{n}$ Dirac structure which vanishes in combination with $N^\sigma = \gamma \not{n}^\sigma$. We can use PSNR to extend this reasoning to all power-suppressed terms with a n^σ Lorentz structure and a Dirac structure that (anti)commutes with $\not{n}$. Considering the emission of arbitrary numbers of gluons from these L -type building blocks, one can exchange the roles of p_{2-} and $k_{n+1,-}$ in the discussion of arbitrary gluon emissions from the equivalent of (3.85). The same mechanism applies, but it can at most maintain a given power of $k_{n+1,-}$ (p_{2-}) and not add powers of $k_{n+1,-}$ (p_{2-}) while adding gluon emissions. Therefore, all terms but the second in (3.95) do not give a double-logarithmic contribution to L -type building blocks with arbitrary numbers of gluon emissions.

The only term in (3.95) that remains to be analyzed is $N^\sigma = \gamma_\perp^\sigma$. As we have just seen all other terms in (3.95) do not contribute because of the additional inverse power of $k_{n+1,-}$ when applied in (3.93). Applying similar reasoning also for $N^\sigma = \gamma_\perp^\sigma$ only $g_{\perp\sigma}^{\mu\sigma}$ can contribute in the gluon propagator in (3.93). This term already comes with the correct powers of $k_{n+1,-}$ and $k_{n+1,+}$ and $k_{n+1,\perp}$ can appear at most linear, so only $m_\eta \bar{u}_0$ can contribute in the spectator-quark propagator. After integrating out k_{n+1} , the result is again of $\Gamma_2(k_{n+2})$ -type (cf. (3.57)) matching our expectation from the previous section (3.64). We stress that this is the only contribution to the initial building block with a three-gluon vertex that we considered in the previous section (cf. (3.64)). For an L -type building block with one gluon emission this contribution is described by



$$\begin{aligned}
&= \frac{i}{k_{n+2,+}k_{n+1,-}} \left(g_{\perp}^{\mu\lambda_1} - \frac{k_{n+1,n+2,\perp}^{\mu} \bar{n}^{\lambda_1} + k_{n+1,n+2,\perp}^{\lambda_1} \bar{n}^{\mu}}{k_{n+1,-}} + \frac{k_{n+2,+}}{k_{n+1,-}} \bar{n}^{\mu} \bar{n}^{\lambda_1} \right) \\
&\quad \left(g_{\lambda_1\lambda_2} (-2k_{n+1} + \dots)^{\rho_1} + g_{\lambda_1}^{\rho_1} (k_{n+1} + \dots)_{\lambda_2} + g_{\lambda_2}^{\rho_1} (k_{n+1} + \dots)_{\lambda_1} \right) \\
&\quad \frac{i}{k_{w+}k_{n+1,-}} \left(g_{\perp}^{\lambda_2}{}_{\sigma} - \frac{k_{n+1,w,\perp}^{\lambda_2} \bar{n}_{\sigma} + k_{n+1,w,\perp,\sigma} \bar{n}^{\lambda_2}}{k_{n+1,-}} + \frac{k_{w+}}{k_{n+1,-}} \bar{n}^{\lambda_2} \bar{n}_{\sigma} \right) i\gamma_{\mu} i(\not{k}_{n+1} + m_{\eta} \bar{u}_0) \frac{\gamma_{\perp}^{\sigma}}{k_{n+1,+}}, \quad (3.96)
\end{aligned}$$

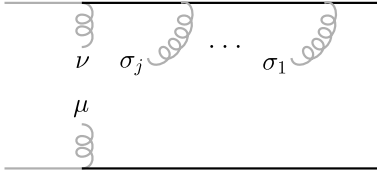
where we omitted the color structure, and the Lorentz index of the emitted gluon is denoted by ρ_1 . Again we have to combine the $g_{\perp}^{\mu\lambda_1}$ and $g_{\perp}^{\lambda_2}{}_{\sigma}$ terms with the first term in the three-gluon vertex to obtain a double-logarithmic contribution. Extrapolating to m gluon emissions and integrating out k_{n+1} , we obtain

$$\propto n^{\rho_1} \dots n^{\rho_m}. \quad (3.97)$$

We are now in a position to combine the outputs and inputs of the H - and L -type building blocks calculated so far to constrain a first subclass of diagrams. There is no need to constrain a H_0 -type building block with no gluon emission because we know that it contributes. It was one of the central objects in the analysis of the previous section (cf. (3.57)). Considering one or more gluon emissions from the gluon line of an H_0 building block there is one contribution that is not power suppressed in (3.88). If it is contracted with an L_1 building block following the H_0 building block this contribution vanishes exactly since $\not{n} = 0$. All other inputs to an L_1 building block in (3.95) are already once power suppressed. All terms, except $\gamma_{\perp}^{\sigma}$, give no contribution as an input to an L_1 building block. If this term is the only emissions of the H_0 building block and the L_1 building block does not emit any gluon itself there is a contribution as expected (cf. (3.64)). If there are multiple emissions from the gluon line of the H_0 building block, the L_1 building block must contract with the first emission of the gluon line of the H_0 building block and an L_2 building block contracts with any of the other emission, then it can be shown that there is again only one inverse power of k_{2+} and we can apply the previous analysis of the L -type building blocks such that also this contribution vanishes. This reasoning can straightforwardly be extended to emissions of L -type building blocks. Hence, we constrained the class of diagrams where the H_0 building block has any number of emission from its gluon line that are absorbed from any subsequent L -type building blocks, including crossings of gluons as long as they do not cross the gluon line of the H_0 building block, since this would imply emission of the H_0 building block from the active-quark line.

We now extend the analysis of the H_0 building block to emissions from the active-quark line. Again we first consider the power-suppressed contribution from the Dirac projector of the η_c meson

$\mathcal{P}_\eta$. The equivalent of $\tilde{H}_{0,0,\mu\nu}$ (cf. (3.83)) for an arbitrary number m of gluon emissions from the active quark-line is then given by



$$\propto \gamma_\mu \gamma_5 \prod_{j=1}^m \left[i \gamma^{\sigma_j} \left(\frac{i(\not{p}_1 - \not{k}_{w_j} + m_\eta u_0)}{k_{w_j+} p_{1-}} \right) \right] \gamma_\nu$$

$$\approx \gamma_\mu \gamma_5 \prod_{j=1}^m \left[-\gamma^{\sigma_j} \left(\frac{\not{n}}{2k_{w_j+}} \right) \right] \gamma_\nu, \quad (3.98)$$

where the order of the emissions is understood by inspecting the indices in the diagram. We also know from the analysis of the gluon line of the H_0 building block without emission from the active-quark line that only terms with $\gamma_\mu \rightarrow \gamma_{\perp\mu}$ and $\gamma_\nu \rightarrow \gamma_{\perp\nu}$ are not further power suppressed. Inserting this into (3.98) we conclude that these terms do not contribute due to PSNR. The equivalent contribution from the part of $\mathcal{P}_\eta$ that is not power suppressed is

$$\propto \gamma_\mu \gamma \not{n} \gamma_5 \prod_{j=1}^m \left[i \gamma^{\sigma_j} \left(\frac{i(\not{p}_1 - \not{k}_{w_j} + m_\eta u_0)}{k_{w_j+} p_{1-}} \right) \right] \gamma_\nu. \quad (3.99)$$

Collecting only terms that receive no power suppression we obtain

$$\propto \gamma_\mu \gamma \not{n} \gamma_5 \prod_{j=1}^m \left[-\gamma^{\sigma_j} \left(\frac{\not{n}}{2k_{w_j+}} \right) \right] \gamma_\nu \approx \gamma_\mu \gamma \not{n} \gamma_5 \prod_{j=1}^m \left[-\left(\frac{n^{\sigma_j}}{k_{w_j+}} \right) \right] \gamma_\nu. \quad (3.100)$$

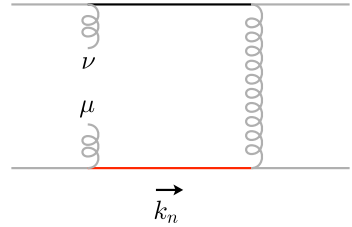
In terms of Dirac structure and power suppression this is precisely $\tilde{H}_{0,1,\mu\nu}$ (cf. (3.83)). Hence, the analysis of the gluon line proceeds exactly analogous. Nevertheless, there are two cases that need special attention before we may conclude that these terms do not contribute. If there is only one emission from the gluon line of the H_0 building block and this emission contracts with a subsequent L_1 building block this contraction does not vanish by itself. A naive topological analysis of a diagram where the L_2 building block connects to the active-quark line of the H_0 building block yields two inverse powers of k_{2+} which would violate the assumption of the discussion of L -type building blocks. However, the only term that contributes (cf. (3.89)) comes with a power of k_{2+} in the numerator, which was omitted in (3.89). The second case is if there are one or more emissions from the active-quark line of an H_0 building block and no emission from the gluon line this generates a term which has the form of the last term in (3.86). This term is beyond the previously considered input of L -type building blocks in (3.95) and, in particular, it does not (anti)commute with a $\not{n}$ Dirac structure. Nevertheless this term has no inverse powers of k_{1+} and a subsequent L_1 building block cannot generate appropriate powers of k_{1+} . Hence, we conclude that there is no contribution from the terms in (3.100). If we instead assume that there is one emission from the active-quark line and one power suppression by not considering the $\not{p}_1$ term in the active-quark propagator, the respective contribution is given by

$$\propto \gamma_{\perp\mu} \gamma \not{n} \gamma_5 i \gamma^{\sigma_1} \left(\frac{i(-\not{k}_w + m_\eta u_0)}{k_{w+} p_{1-}} \right) \gamma_{\perp\nu} \simeq -\frac{\gamma}{k_{w+} p_{1-}} \gamma_{\perp\mu} (2n^{\sigma_1} \not{k}_{w\perp} + 2m_\eta u_0 n^{\sigma_1} - 2k_{w+} \gamma_{\perp}^{\sigma_1}) \gamma_{\perp\nu} \gamma_5, \quad (3.101)$$

where we again anticipated that the contraction of μ and ν is in the $\perp$ -direction. The first two terms on the r.h.s of (3.101) do not contribute for well-understood reasons. The third term vanishes upon

contraction of μ and ν . Extending this argument to arbitrary numbers of gluon emissions from the active-quark line, we notice that the power suppressed propagator must be the one closest to $\gamma_{\perp\nu}$ since otherwise there is an additional power suppression due to PSNR. Simplifying the Dirac structure as in (3.101) proceeds analogous and we conclude that there is no contribution from these terms. This concludes the discussion of emissions from the active-quark line of the H_0 building block and allows us to extend the class of constrained diagrams to those with crossings of the gluon line of the H_0 building block.

We now extend our analysis of H -type building blocks to cases where lower loops were built up by iteration steps well-known from the previous section. In other words, we analyze an H_n building block whose input has the form of the Dirac structures $\Gamma_i(k_n)$ (cf. (3.57) and (3.58)). In analogy to the discussion of the H_0 building block, we first focus on the case with no gluon emission from the active-quark line. Since the gluon line is independent of external collinear momenta, here most of the constraints will derive from restricting to appropriate powers of k_{n+} and k_{n-} . Similar to the discussion of H_0 we simplify the Dirac structure before we contract with the gluon line. Assuming $\Gamma_i(k_n)$ as input, we obtain the Dirac structures



$$\propto \tilde{H}_{n,i,\mu\nu}, \quad (3.102)$$

where

$$\tilde{H}_{n,i,\mu\nu} \in \left\{ \frac{\gamma_\mu \not{n}_\perp \gamma_\nu \gamma_5}{k_{n+}^2}, \frac{\gamma_\mu \gamma_\nu \gamma_5}{k_{n+}}, \frac{\gamma_\mu \not{n}_\perp \gamma_\nu \gamma_5}{k_{n+}} k_{n-}, \frac{\gamma_\mu \not{k}_{n+} \not{n}_\perp \gamma_\nu \gamma_5}{k_{n+}^2}, \frac{\gamma_\mu \not{k}_{n+} \gamma_\perp \gamma_\nu \gamma_5}{k_{n+}}, \frac{\gamma_\mu \not{n}_\perp \gamma_\nu \gamma_5}{k_{n+}}, \frac{\gamma_\mu \not{k}_{n+} \not{n}_\perp \gamma_\nu \gamma_5}{k_{n+}} \right\}, \quad (3.103)$$

which are all once power suppressed in γ as expected due to PSNR. Keeping track of which $\Gamma_i(k_n)$ contributes to a given $\tilde{H}_{n,i,\mu\nu}$ yields

$$\begin{aligned} \Gamma_1(k_n) &\rightarrow \tilde{H}_{n,1,\mu\nu}, \tilde{H}_{n,2,\mu\nu}, \tilde{H}_{n,3,\mu\nu}, \tilde{H}_{n,4,\mu\nu}, \tilde{H}_{n,5,\mu\nu}, \tilde{H}_{n,6,\mu\nu}, \tilde{H}_{n,7,\mu\nu}, \\ \Gamma_2(k_n) &\rightarrow \tilde{H}_{n,1,\mu\nu}, \tilde{H}_{n,4,\mu\nu}, \tilde{H}_{n,6,\mu\nu}, \\ \Gamma_3(k_n) &\rightarrow \tilde{H}_{n,1,\mu\nu}, \tilde{H}_{n,3,\mu\nu}, \tilde{H}_{n,4,\mu\nu}, \tilde{H}_{n,5,\mu\nu}, \tilde{H}_{n,6,\mu\nu}, \tilde{H}_{n,7,\mu\nu}. \end{aligned} \quad (3.104)$$

We note that it matches with our expectation from the iteration equation (3.66) that $\Gamma_1(k_n)$ and $\Gamma_3(k_n)$ give a richer structure than $\Gamma_2(k_n)$. Additionally, we conclude from (3.66), (3.61) and (3.104) that while all $\tilde{H}_{n,i,\mu\nu}$ may contribute for no emission from the gluon line, none of $\tilde{H}_{n,1,\mu\nu}$, $\tilde{H}_{n,4,\mu\nu}$ or $\tilde{H}_{n,6,\mu\nu}$ should contribute to single gluon emission from the gluon line.

We now start to analyze the contributions from all $\tilde{H}_{n,i,\mu\nu}$ in (3.103) one-by-one and begin with $\tilde{H}_{n,1,\mu\nu}$. Applying the LCG gluon propagator in (3.82) (with $k_w \rightarrow k_{n+1}$) to $\tilde{H}_{n,1,\mu\nu}$ corresponds to a gluon line with no emission. There has to be a power of k_{n+} in the numerator to obtain a double logarithm. But there is no k_{n+} directly and at most a linear term in $k_{n\perp}$, so there cannot be a contribution from this term. Considering one gluon emission from the gluon line we have to analyze

$$\frac{i}{k_{n+1,+} k_{n-}} \left(g_{\perp}^{\mu\lambda_1} - \frac{k_{n,n+1,\perp}^{\mu} \bar{n}^{\lambda_1} + k_{n,n+1,\perp}^{\lambda_1} \bar{n}^{\mu}}{k_{n-}} + \frac{k_{n+1,+}}{k_{n-}} \bar{n}^{\mu} \bar{n}^{\lambda_1} \right) \left(g_{\lambda_1\lambda_2} (2k_n + \dots)^{\rho_1} + g_{\lambda_1}^{\rho_1} (-k_n + \dots)_{\lambda_2} \right)$$

$$+ g_{\lambda_2}^{\rho_1} (-k_n + \dots)_{\lambda_1} \left(\frac{i}{k_{w+} k_{n-}} \left(g_{\perp}^{\lambda_2 \nu} - \frac{k_{n,w,\perp}^{\lambda_2} \bar{n}^\nu + k_{n,w,\perp}^\nu \bar{n}^{\lambda_2}}{k_{n-}} + \frac{k_{w+}}{k_{n-}} \bar{n}^{\lambda_2} \bar{n}^\nu \right) \right), \quad (3.105)$$

where ρ_1 is the open Lorentz index of the three-gluon vertex that emits the gluon. Due to the definition of $\tilde{H}_{n,1,\mu\nu}$ (cf. (3.103)) and PSNR, the ν -index cannot be in the $\perp$ -direction. Hence, there are at least three inverse powers of k_{n-} . But there is no combination of terms in the numerator that generates two powers of k_{n-} . It is straightforward to extend this reasoning to arbitrary numbers of emissions from the gluon line and, therefore, there is no double-logarithmic contribution from $\tilde{H}_{n,1,\mu\nu}$.

Turning to $\tilde{H}_{n,2,\mu\nu}$ we obtain a contribution from the $g_{\perp}^{\mu\nu}$ -term of the LCG gluon propagator in (3.82) for no emission from the gluon line and no other term in (3.82) can contribute. The respective contribution is a $\Gamma_2(k_{n+1})$ Dirac structure, i. e. an identity in Dirac space with one inverse power of $k_{n+1,+}$. To analyze a single gluon emission from the gluon line we again have to contract with (3.105). The only terms that can possibly generate appropriate powers of k_{n-} are the $g_{\perp}^{\mu\nu}$ -terms in the gluon propagators combined with

$$g_{\lambda_1 \lambda_2} (2k_n + \dots)^{\rho_1} \rightarrow g_{\perp, \lambda_1 \lambda_2} k_{n-} n^{\rho_1}, \quad (3.106)$$

in the three-gluon vertex. This leads to a contribution of the form n^{ρ_1} , i. e. identity in Dirac space. Generalizing to j emissions from the gluon line, the contribution is proportional to $n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$. Up to multiplicative factors of $n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$, this is not only an identity in Dirac space but also again a $\Gamma_2(k_{n+1})$ Dirac structure.

There are two kinds of contributions from $\tilde{H}_{n,3,\mu\nu}$, which makes the discussion more involved. Similar to $\tilde{H}_{n,1,\mu\nu}$ the contraction with ν cannot be in the $\perp$ -direction due to PSNR. For no gluon emission we contract with the LCG gluon propagator (3.82) and see that the numerator should be independent of k_{n+} and k_{n-} . There are contributions from two terms: $k_{n+1,\perp}^\mu \bar{n}^\nu$ and $k_{n+1,+} n^\mu \bar{n}^\nu$ which together correspond to a $\Gamma_3(k_{n+1})$ Dirac structure (cf. (3.58)). For one or more gluon emissions we contract with (3.105) or its equivalent with more emissions. In the gluon propagator closest to the active-quark line, i. e. with Lorentz index ν , we have to choose terms with additional inverse powers of k_{n-} due to PSNR. Therefore in any other gluon propagator only the $g_{\perp}$ -term can contribute, because any other term leads to additional inverse powers of k_{n-} which cannot be canceled by the numerator. The term $\bar{n}^{\lambda_2} \bar{n}^\nu$ in the first gluon propagator has to be combined with the term $g_{\lambda_1}^{\rho_1} k_{n,\lambda_2}$ of the three gluon vertex to give a contribution of the form $\gamma_{\perp}^{\rho_1}$. For more emissions this contribution generalizes to $\gamma_{\perp}^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$, where ρ_1 is the Lorentz index of the first gluon emission starting from the active-quark line. Likewise, the term $k_{n,w,\perp}^{\lambda_2} \bar{n}^\nu$ combines with $g_{\lambda_1 \lambda_2} k_n^{\rho_1}$ to the contribution $k_{w\perp} n^{\rho_1}$, which generalizes to $k_{w\perp} n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$, where k_w is a placeholder for the sum of loop momenta—omitting k_n —of the gluon propagator attaching to the active-quark line.

Turning to $\tilde{H}_{n,4,\mu\nu}$ and no gluon emission only the term $k_{n\perp}^\mu \bar{n}^\nu$ contributes and the result is a $\Gamma_2(k_{n+1})$ Dirac structure. For one gluon emission we contract with (3.105). Even if we consider the $g_{\perp}^{\mu\lambda_1}$ -term in the second propagator there are two inverse powers of k_{n+} and three inverse powers of k_{n-} . Hence, only a term with $k_{n\perp}^2$ and k_{n-} in the numerator can generate double-logarithmic powers of k_{n+} and k_{n-} . The only combination of terms that achieves this is $g_{\perp}^{\mu\lambda_1}$, $g_{\lambda_1 \lambda_2} k_n^{\rho_1}$ and $k_{n\perp}^{\lambda_2} \bar{n}^\nu$, leading to an identity in Dirac space and a n^{ρ_1} Lorentz structure. This reasoning straightforwardly generalizes to more emissions and the result is again an identity in Dirac space and a $n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$ Lorentz structure.

Extending this kind of analysis to $\tilde{H}_{n,5,\mu\nu}$ we find that there is no contribution for no emission, one emission or more emissions. The reason is that $k_{n\perp}$ must combine with another $k_{n\perp}$ since a

$\tilde{H}_{n,i,\mu\nu}$	no emission	one emission	multiple emissions
$\tilde{H}_{n,1,\mu\nu}$	—	—	—
$\tilde{H}_{n,2,\mu\nu}$	$\Gamma_2(k_{n+1})$	n^{ρ_1}	$n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$
$\tilde{H}_{n,3,\mu\nu}$	$\Gamma_3(k_{n+1})$	$\gamma_\perp^{\rho_1}, \not{k}_{n+1,\perp} n^{\rho_1}$	$\gamma_\perp^{\rho_1} n^{\rho_2} \dots n^{\rho_j}, \not{k}_{w,\perp} n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$
$\tilde{H}_{n,4,\mu\nu}$	$\Gamma_2(k_{n+1})$	n^{ρ_1}	$n^{\rho_1} n^{\rho_2} \dots n^{\rho_j}$
$\tilde{H}_{n,5,\mu\nu}$	—	—	—
$\tilde{H}_{n,6,\mu\nu}$	—	—	—
$\tilde{H}_{n,7,\mu\nu}$	—	—	—

Table 3.1: Contributions from the different Dirac structures $\tilde{H}_{n,i,\mu\nu}$ (cf. (3.103)) to emissions from the gluon line of an H_n building block with no emissions from the active-quark line. For one or more emissions we keep the dependence on higher loop momenta implicit if it does not contract to a Dirac structure.

linear $k_{n,\perp}$ vanishes, but then only the mass term of the on-shell constraint can contribute because there is only one inverse power of k_{n+} . With these restrictions in place it is not possible to find a term that gives appropriate powers of k_{n-} .

Contracting the Lorentz index ν of $\tilde{H}_{n,6,\mu\nu}$ in the $\perp$ -direction gives no contribution due to PSNR. The remaining analysis proceeds similar to that of $\tilde{H}_{n,2,\mu\nu}$. Hence, there is no term that can generate appropriate powers of k_{n-} and there is no contribution from $\tilde{H}_{n,6,\mu\nu}$.

The analysis of $\tilde{H}_{n,7,\mu\nu}$ proceeds exactly analogous to the one of $\tilde{H}_{n,5,\mu\nu}$ and, hence, there is no contribution.

A summary of the outputs of the H_n building block with emissions from the gluon line can be found in Table 3.1. The results match with the expectation from the previous section. For concreteness, combining Table 3.1 with the relations between $\Gamma_i(k_n)$ and $\tilde{H}_{n,j,\mu\nu}$ in (3.104), and the above discussion of L -type building blocks implies that $\Gamma_2(k_n)$ Dirac structures yield exclusively $\Gamma_2(k_{n+1})$, $\Gamma_3(k_n)$ Dirac structures mix into $\Gamma_2(k_{n+1})$ and themselves, $\Gamma_1(k_n)$ Dirac structures mix into $\Gamma_2(k_{n+1})$ and $\Gamma_3(k_{n+1})$, and only $\Gamma_1(k_n)$ and $\Gamma_3(k_n)$ contribute to diagrams with one gluon emission.

We now extend the analysis of the H_n building block to emissions from the active-quark line. To this end, we have to make appropriate changes to the Dirac structure that lead to the $\tilde{H}_{n,j,\mu\nu}$ (cf. (3.102)) to account for the additional gluon emissions from the active-quark line. The modification amounts to

$$\begin{aligned}
& i\gamma_\mu i(-\not{k}_n + m_\eta \bar{u}_0) \frac{1}{k_{n+}} \sum_{i=1}^3 c_i^{(n-1)} \Gamma_i(k_n) \gamma_5 \frac{i(\not{\phi}_\eta - \not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} i\gamma_\nu \\
\rightarrow & i\gamma_\mu i(-\not{k}_n + m_\eta \bar{u}_0) \frac{1}{k_{n+}} \sum_{i=1}^3 c_i^{(n-1)} \Gamma_i(k_n) \gamma_5 \frac{i(\not{\phi}_\eta - \not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} \prod_{j=1}^m \left[i\gamma^{\sigma_j} \frac{i(\not{\phi}_\eta - \not{k}_{w_j} + m_\eta u_0)}{p_\eta - k_{w_j+}} \right] i\gamma_\nu,
\end{aligned} \tag{3.107}$$

where we expanded the denominators in the hierarchy of the strong-rapidity ordering (3.53). For $\Gamma_2(k_n)$ and $\Gamma_3(k_n)$, which are already power suppressed, it is straightforward to understand the

necessary modification. Even without the gluon emission the first propagator has to be power expanded according to

$$\frac{i(\not{p}_\eta - \not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} \approx i \frac{\not{p}_\eta}{2k_{n+}}. \quad (3.108)$$

Expanding also the additional propagators we obtain

$$\begin{aligned} \frac{i(\not{p}_\eta - \not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} \prod_{j=1}^m \left[i\gamma^{\sigma_j} \frac{i(\not{p}_\eta - \not{k}_{w_j} + m_\eta u_0)}{p_\eta - k_{w_j+}} \right] &\approx i \frac{\not{p}_\eta}{2k_{n+}} \prod_{j=1}^m \left[-\gamma^{\sigma_j} \frac{\not{p}_\eta}{2k_{w_j+}} \right] \\ &= i \frac{\not{p}_\eta}{2k_{n+}} \prod_{j=1}^m \left[-\frac{n^{\sigma_j}}{k_{w_j+}} \right], \end{aligned} \quad (3.109)$$

which has the same Dirac structure and power of k_{n+} as (3.108). Hence, these terms do not generate additional $\tilde{H}_{n,j,\mu\nu}$ and the corresponding results in Table 3.1 only have to be modified by a factor

$$\prod_{j=1}^m \left[-\frac{n^{\sigma_j}}{k_{w_j+}} \right]. \quad (3.110)$$

For $\Gamma_1(k_n) = \gamma \not{p}_\eta$, on the other hand, the term $\not{p}_\eta \approx p_\eta - \not{p}/2$ in the first propagator cancels out which leads to a power suppression such that in all remaining propagators only the leading-power terms contribute which gives

$$\begin{aligned} \tilde{H}_{n,\Gamma_1,\mu\nu}^{\sigma_1 \dots \sigma_m} &= i\gamma_\mu i(-\not{k}_n + m_\eta \bar{u}_0) \frac{\gamma \not{p}_\eta \gamma_5}{k_{n+}} \frac{i(\not{p}_\eta - \not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} \prod_{j=1}^m \left[i\gamma^{\sigma_j} \frac{i(\not{p}_\eta - \not{k}_{w_j} + m_\eta u_0)}{p_\eta - k_{w_j+}} \right] i\gamma_\nu \\ &\approx i\gamma_\mu i(-\not{k}_n + m_\eta \bar{u}_0) \frac{\gamma \not{p}_\eta \gamma_5}{k_{n+}} \frac{i(-\not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} \left(-\gamma^{\sigma_1} \frac{\not{p}_\eta}{2k_{w_1+}} \right) \prod_{j=2}^m \left[-\frac{n^{\sigma_j}}{k_{w_j+}} \right] i\gamma_\nu. \end{aligned} \quad (3.111)$$

We anticipate that the $\perp$ -component of γ_ν does not contribute due to PSNR, which leads to

$$\tilde{H}_{n,\Gamma_1,\mu\nu}^{\sigma_1 \dots \sigma_m} \approx i\gamma_\mu i(-\not{k}_n + m_\eta \bar{u}_0) \frac{\gamma \not{p}_\eta \gamma_5}{k_{n+}} \frac{i(-\not{k}_n + m_\eta u_0)}{p_\eta - k_{n+}} \left(-\frac{\gamma^{\sigma_1}}{k_{w_1+}} \right) \prod_{j=2}^m \left[-\frac{n^{\sigma_j}}{k_{w_j+}} \right] i n_\nu. \quad (3.112)$$

After further simplifying the Dirac structure in (3.112) most terms can be related to $\tilde{H}_{n,i,\mu\nu}$ multiplied by factors as in (3.110), or they do not contribute because of missing inverse powers of k_{n+} or PSNR. Focusing for simplicity on single-gluon emission from the active-quark line, the Dirac structures that need additional attention are

$$\tilde{H}_{n,\Gamma_1,i,\mu\nu}^{\sigma_1} \in \left\{ \frac{\gamma_\mu \not{k}_{n\perp} \gamma_\perp^{\sigma_1} n_\nu \gamma_5}{k_{n+}}, \frac{\gamma_\mu \gamma_\perp^{\sigma_1} n_\nu \gamma_5}{k_{n+}} \right\}. \quad (3.113)$$

In analogy to the above discussion of contributions from gluon lines attached to $\tilde{H}_{n,i,\mu\nu}$, it is straightforward to see that both terms in (3.113) cannot generate appropriate powers of k_{n+} and k_{n-} . Combining the possible outputs of an H_n building block that has the $\Gamma_i(k_n)$ Dirac structures as input with the above discussion of L -type building blocks expands the class of constrained diagrams. We are now able to constrain any H -type building block that has subsequent L -type building blocks that absorb all emissions of the H -type building block or intermediate gluon emissions from L -type building blocks, except the contributing diagrams we discussed in Sec. 3.2.1. An example diagram

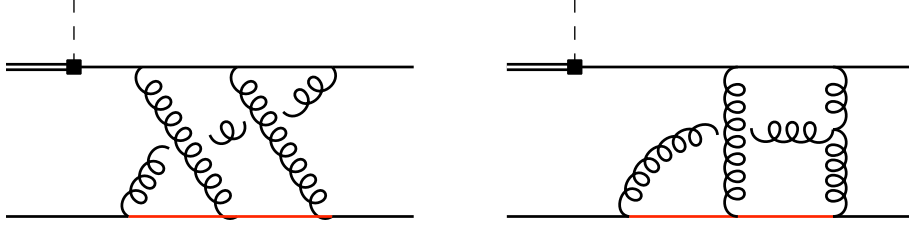


Figure 3.17: Sample soft-quark diagrams where an gluon emission of an H -type building block is not absorbed by L -type building blocks before the next H -type building block. The diagram in the left (right) panel corresponds to the sequence of building blocks $L_{2,\sigma} H_1^{\dot{\sigma}} H_0^{\sigma}$ ($L_{2,\sigma} H_1^{\dot{\sigma}} H_0^{\sigma_i}$).

of this class can be found in Fig. 3.15. We note again that diagrams with four-gluon vertices are constrained since the momentum dependence is required in all cases to obtain double logarithms at leading power and a four-gluon vertex is independent of momenta.

There is only one subclass of diagrams that is left to constrain. These are soft-quark diagrams where not all gluon emissions of H -type building blocks are absorbed before the next H -type building block. Examples of this class of diagrams are shown in Fig. 3.17. Again we start with an analysis of emissions of an H_0 building block. We know from the previous discussions that any H_0 building block followed by two or more L -type building blocks gives no contribution at the double-logarithmic level and at leading power. Hence, we only have to consider the cases $\dots H_1 H_0$ with one or more emissions from the H_0 building block and $\dots H_2 L_1 H_0$ with two or more emissions from the H_0 building block.⁶ From the discussion of the H_0 building block we know that the input of a sequence $L_1 H_0$ to H_2 generates no new structures with respect to the input of an H_0 building block to an H_1 building block for arbitrary gluon emissions of the H_0 and L_1 building blocks. For notational simplicity we therefore discuss only the case $\dots H_1 H_0$. Terms that enter the H_1 building block have the form

$$H_{0,i}^{(\rho_1)[\rho_2 \dots \rho_m]} \in \left\{ \gamma \frac{\not{k}_5}{k_{1+} p_{2-}}, \gamma \frac{\gamma_5}{k_{1+} p_{2-}^2}, \gamma \frac{\not{k}_{1\perp} \gamma_5}{k_{1+} p_{2-}^2}, \gamma \frac{\gamma^{\rho_1} \gamma_5}{k_{1+} p_{2-}^2}, \gamma \frac{\not{k}_{w\perp} \gamma_5}{k_{1+} p_{2-}^2} \right\}, \quad (3.114)$$

where the first three terms correspond to $\Gamma_i(k_1)$ Dirac structures and the last two terms are genuinely new structures generated by gluon emissions. All terms in (3.114) are understood to be multiplied by a factor

$$\prod_{j=2}^m \left[-\frac{n^{\rho_j}}{k_{w_j+}} \right] \quad (3.115)$$

to account for any number of gluon emissions as indicated by the index $[\rho_2 \dots \rho_m]$ in (3.114). The details of this factor are irrelevant for the following analysis and we keep it implicit. Even if the input is the same in case of the terms that resemble $\Gamma_i(k_1)$, the analysis that lead to the Dirac structures $\tilde{H}_{n,i,\mu\nu}$ in (3.103) has to be repeated. The reason is that in case of a crossing of an L -type building block to an H_0 building block no active-quark propagator is dominated by k_{1+} . We note that this is also the reason that there is no need to consider the last term in (3.86): It cannot generate appropriate powers of k_{1+} . Simplifying the Dirac structure of the H_1 building block without emissions from the active-quark line leads to the following equivalents of $\tilde{H}_{n,i,\mu\nu}$ (cf. (3.103))

⁶ The unabsorbed gluon emission can also originate from the L_1 building block in which case both the H_0 and L_1 building block have one or more emissions.

$$\tilde{H}_{1,i,\mu\nu}^{(\rho_1)[\rho_2\cdots\rho_m]} \in \left\{ \frac{\gamma_\mu \not{n} \not{\gamma}_\nu \gamma_5}{k_{1+}}, \frac{\gamma_\mu \not{k}_{1\perp} \gamma_\perp \gamma_5}{k_{1+}}, \frac{\gamma_\mu \not{k}_{w\perp} n_\nu \gamma_5}{k_{1+}}, \frac{\gamma_\mu \not{k}_{1\perp} n_\nu \gamma_5}{k_{1+}}, \frac{\gamma_\mu \not{\gamma}_\nu \gamma_5}{k_{1+}}, \right. \\ \left. \frac{\gamma_\mu \not{k}_{1\perp} \not{k}_{w\perp} n_\nu \gamma_5}{k_{1+}}, \frac{\gamma_\mu \not{k}_{1\perp} \gamma_\perp^{\rho_1} n_\nu \gamma_5}{k_{1+}}, \frac{\gamma_\mu \gamma_\perp^{\rho_1} n_\nu \gamma_5}{k_{1+}} \right\}. \quad (3.116)$$

In analogy to the discussion of gluon lines attaching to $\tilde{H}_{n,i,\mu\nu}$ we find that none of the Dirac structure in (3.116) except the first lead to a contribution upon attaching gluon lines. In fact, $\tilde{H}_{1,1,\mu\nu}^{(\rho_1)[\rho_2\cdots\rho_m]}$ can be related to $\tilde{H}_{n,2,\mu\nu}$ and $\tilde{H}_{n,6,\mu\nu}$ (cf. (3.103)) up to the factor in (3.115). Hence, the result is also an identity in Dirac space and has the Lorentz structure $n^{\rho_1} \dots n^{\rho_m}$, which resembles $H_{0,2}^{(\rho_1)[\rho_2\cdots\rho_m]}$ (cf. (3.114)). We note that $\tilde{H}_{1,1,\mu\nu}^{(\rho_1)[\rho_2\cdots\rho_m]}$ exclusively originates from $H_{0,1}^{(\rho_1)[\rho_2\cdots\rho_m]}$. Furthermore, an analysis of the Dirac structure in the presence of gluon emissions from the active-quark line in analogy to the discussion of (3.107) does not generate new structures with respect to (3.116). Hence, there is no need to consider this case separately. Coming back to the discussion of the output of $\tilde{H}_{1,1,\mu\nu}^{(\rho_1)[\rho_2\cdots\rho_m]}$ there are two options. Since by definition of the case under consideration, there are unabsorbed gluon emissions from previous building blocks there has to be at least one additional L -type building block to absorb the gluons. There can be another H -type building block before the required L -type building block leading to the sequence $\dots H_2 H_1 H_0$ with unabsorbed gluon emissions from H_0 and potentially also from H_1 . Since we know that the output of $\dots H_1 H_0$ must be of the form $H_{0,2}^{(\rho_1)[\rho_2\cdots\rho_m]}$ (cf. (3.114)) H_2 cannot lead to double-logarithmic contributions. The other option is that $\dots H_1 H_0$ is followed by an L_2 building block leading to the sequence $\dots L_2 H_1 H_0$. Again we know that the output of $\dots H_1 H_0$ must be of the form $H_{0,2}^{(\rho_1)[\rho_2\cdots\rho_m]}$ (cf. (3.114)), which has a Dirac structure that (anti)commutes with $\not{n}$ and a Lorentz structure $n^{\rho_1} \dots n^{\rho_m}$. Additionally, an analysis of the topologies of all possible graphs similar to that in Fig. 3.16 implies that there is at most one inverse power k_{2+} . According of the above discussion of L -type building blocks there cannot be a double-logarithmic contribution from L_2 .

The only case that is left to constrain is an H_n building block with $\Gamma_i(k_n)$ as input and where not all emitted gluons are absorbed before the next H -type building block. Inspecting the output of an H_n building block in Table 3.1 and the corresponding discussion of emission from the active-quark line of H_n , we notice that there are no new structures with respect to (3.114). Hence, there is no need to discuss this case separately. In fact, the discussion is simpler since the equivalent of $H_{0,1}^{(\rho_1)[\rho_2\cdots\rho_m]}$ is missing in this case.

Finally, having discussed all possible classes of diagrams, we conclude that any soft-quark diagram except those discussed in Sec. 3.2.1 cannot give a double-logarithmic contribution at leading power. In addition, we revealed the detailed mechanism of relevant diagrams that may be helpful for an EFT formulation of the problem.

3.3 SOFT-GLUON CORRECTIONS

The aim of this section is to combine soft-quark diagrams with soft-gluon effects in a generic way that can be extended to all orders. In the discussion of the mixed soft-gluon and soft-quark corrections in Sec. 3.1.3 we noticed that for some diagrams the Dirac structure only factorized in the sum of certain diagrams. It is important to understand this non-factorization in detail, since even slight modifications of the Dirac structure might necessitate to repeat or extend the discussion of the previous section. To this end, we remember that only diagrams with a soft gluon attaching to the active-quark propagator of the one-loop box did not factorize. From the analysis of general

soft-quark diagrams we also know that this propagator is the only one that can still receive a power suppression from a $\Gamma_1(k_1)$ Dirac structure. If a term has already received a power suppression, the numerator of the respective active-quark line with an attaching soft gluon has the form

$$\left(p_{\eta-} \frac{\not{p}}{2} + \dots\right) \gamma^\mu \left(p_{\eta-} \frac{\not{p}}{2} + \dots\right), \quad (3.117)$$

where γ^μ attaches to the soft gluon and dots indicate power-suppressed terms of the quark propagators. Upon commuting γ^μ with the active-quark propagator to its right, this becomes

$$\left(p_{\eta-} \frac{\not{p}}{2} + \dots\right) \left(-p_{\eta-} \frac{\not{p}}{2} \gamma^\mu + p_{\eta-} n^\mu + \dots\right). \quad (3.118)$$

The term $\not{p} \gamma^\mu$ vanishes since the other propagator obeys the leading-power equations-of-motion (EOM). The remaining term gives a soft-gluon attachment that factorizes in Dirac space and leads to the usual eikonal for soft-gluon emissions. On the other hand, if there is a $\Gamma_1(k_1)$ Dirac structure present—such as in diagram D'_{10} in Fig. 3.10—the relevant terms read

$$\gamma \not{p} (-\not{k} + m_\eta u_0 + \dots) \left(-p_{\eta-} \frac{\not{p}}{2} \gamma^\mu + p_{\eta-} n^\mu + \dots\right), \quad (3.119)$$

where the term $\not{p} \gamma^\mu$ in the last propagator leads to a contribution that violates factorization in Dirac space. This additional term is precisely canceled by diagrams such as D'_{11} and D'_{12} in Fig. 3.10 restoring factorization in Dirac space. Since this effect relies on the presence of a $\Gamma_1(k_1)$ Dirac structure, it can only occur in a soft-gluon correction to the first box after the η_c meson and even in this case there are factorization-restoring diagrams.

To summarize the above discussion, we now understand why only the soft-gluon corrections to the first box after the η_c meson factorize only in the sum of diagrams. For all other soft-gluon diagrams the Dirac structure factorizes at the level of single diagrams to the respective Born diagram. Hence, we recover the expected property that soft gluons are spin-independent and do not modify the structure of soft-quark diagrams. In fact, it is well-known that soft-gluon corrections exponentiate and the argument of the exponential is given by the quadratic Casimir of the emitting parton multiplied by an appropriate double logarithm. The fact that soft-gluon corrections factorize also allows us to simply determine them by examining the available phase space that eikonalizes them as we did in Sec. 3.1.3. When we discussed soft-gluon corrections to soft-quark diagrams at fixed order, we noticed that the phase space of the logarithm of a given diagram can straightforwardly be identified by inspecting the soft-gluon corrections in the large- N_c limit. This was the case both for corrections to Abelian ladder rungs (cf. Fig. 3.11) and rungs with three-gluon vertices (cf. Fig. 3.8). To understand the general structure of the soft-gluon corrections it is therefore sufficient to analyze how the building block of the contributing soft-quark diagrams are modified by soft-gluon corrections in the large- N_c limit. The relevant diagram for a soft-gluon correction to a generic box is shown in Fig. 3.18. To obtain a double logarithm from the soft gluon, there has to be exactly one inverse power for q_+ and q_- . The momenta of the active-quark propagator in the n -direction are dominated by $p_{\eta-}$. Hence, the relevant propagators are eikonalized by imposing the constraints

$$\begin{aligned} (k_{n+1} - k_n - q)^2 &\approx -k_{n+1,+} q_- &\Rightarrow & q_- \gg k_{n-}, & q_+ \ll k_{n+1,+}, \\ (p_\eta - k_n - q)^2 &\approx -p_{\eta-} q_+ &\Rightarrow & q_+ \gg k_{n+}, & q_- \ll p_{\eta-}, \end{aligned} \quad (3.120)$$

which translates into

$$-\frac{\alpha_s C_F}{2\pi} \int_{k_{n+}}^{k_{n+1,+}} \frac{dq_+}{q_+} \int_{k_{n-}}^{p_{\eta-}} \frac{dq_-}{q_-} = -\frac{\alpha_s C_F}{2\pi} \ln\left(\frac{k_{n+1,+}}{k_{n+}}\right) \ln\left(\frac{p_{\eta-}}{k_{n-}}\right). \quad (3.121)$$

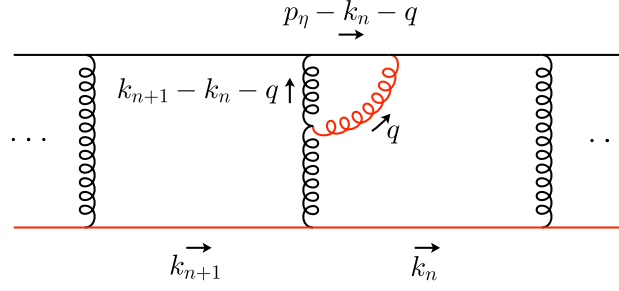


Figure 3.18: Planar soft-gluon attachment to a generic rung in the soft-quark ladder.

Likewise, the soft-gluon correction to the non-Abelian diagram shown in Fig. 3.19 is eikonalized by imposing

$$\begin{aligned} (k_{n+1} - k_n - q)^2 &\approx -k_{n+1,+}q_- &\Rightarrow & q_- \gg k_{n-}, & q_+ \ll k_{n+1,+}, \\ (k_{n-1} - k_n - q)^2 &\approx -k_{n-1,-}q_+ &\Rightarrow & q_+ \gg k_{n+}, & q_- \gg k_{n-1,-}, \end{aligned} \quad (3.122)$$

which leads to the integral

$$-\frac{\alpha_s C_F}{2\pi} \int_{k_{n+}}^{k_{n+1,+}} \frac{dq_+}{q_+} \int_{k_{n-}}^{k_{n-1,-}} \frac{dq_-}{q_-} = -\frac{\alpha_s C_F}{2\pi} \ln \left(\frac{k_{n+1,+}}{k_{n+}} \right) \ln \left(\frac{k_{n-1,-}}{k_{n-}} \right). \quad (3.123)$$

We observe that the soft-gluon corrections to the Abelian and non-Abelian ladder have a similar dependence on $k_{n\pm}$ and $k_{n+1,+}$. On the other hand, the upper limit in the n -direction is $p_{\eta-}$ for the box and $k_{n-1,-}$ for the non-Abelian soft-quark diagram. At this point, it is important to remember that the non-Abelian soft-quark diagram is a two-loop building and both loops are genuinely different from the Abelian box. Hence, there is another diagram that contributes to the soft-gluon corrections of the two-loop non-Abelian building block that gives the corrections corresponding to the loop momentum k_{n-1} (cf. Fig. 3.19). This diagram is given by the equivalent of the diagram in Fig. 3.18 applied to the loop with momentum k_{n-1} in Fig. 3.19 such that the soft gluon attaches to the gluon with momentum $k_{n+1} - k_{n-1}$ and the active quark. The analysis proceeds analogous to the one of the soft-gluon correction of the box (cf. (3.120) and (3.121)) by replacing k_n with k_{n-1} . Considering the sum of the result of this diagram and the diagram in Fig. 3.19, we obtain

$$\begin{aligned} &-\frac{\alpha_s C_F}{2\pi} \left\{ \underbrace{\ln \left(\frac{k_{n+1,+}}{k_{n+}} \right) \ln \left(\frac{k_{n-1,-}}{k_{n-}} \right)}_{\text{non-Abelian insertion}} + \underbrace{\ln \left(\frac{k_{n+1,+}}{k_{n-1,+}} \right) \ln \left(\frac{p_{\eta-}}{k_{n-1,-}} \right)}_{\text{special ladder insertion}} \right\} \\ &= -\frac{\alpha_s C_F}{2\pi} \left\{ \ln \left(\frac{k_{n+1,+}}{k_{n+}} \right) \ln \left(\frac{p_{\eta-}}{k_{n-}} \right) + \ln \left(\frac{k_{n+}}{k_{n-1,+}} \right) \ln \left(\frac{p_{\eta-}}{k_{n-1,-}} \right) \right\}. \end{aligned} \quad (3.124)$$

Therefore both loops of the two-loop non-Abelian building block receive soft-gluon corrections equivalent to those of the Abelian box (cf. (3.121)). Inspecting (3.121) with $k_n \rightarrow p_{\eta}$ we see that there is no equivalent of the soft-gluon correction to the box attaching to the external leg, because there is no available phase space. Similarly, if the three-gluon vertex is the first interaction next to the η_c meson—in the language of the previous section this corresponds to $L_{1,\sigma} H_0^\sigma$ —we have to replace k_{n-1} with p_2 in (3.123), which recovers the equivalent of (3.121) at the double-logarithmic level.

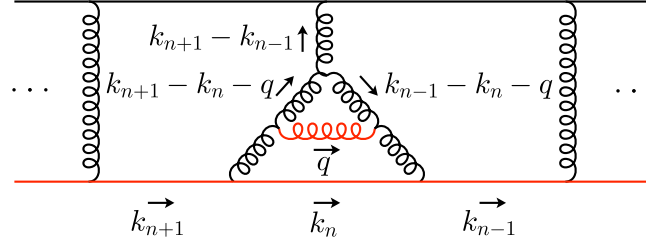


Figure 3.19: Planar soft-gluon attachment to a generic non-Abelian loop in the soft-quark ladder.

We conclude that the appropriate soft-gluon logarithm for any soft-quark loop with momentum k_n , Abelian or non-Abelian and at any position in the ladder, is

$$-\frac{\alpha_s C_F}{2\pi} \ln \left(\frac{k_{n+1,+}}{k_{n+}} \right) \ln \left(\frac{p_{\eta-}}{k_{n-}} \right). \quad (3.125)$$

Then, the next step is to implement this insight into the integral equations of the soft-quarks. In terms of notation this implies that we use q to indicate k_{n+1} or ℓ as the integral equations defined by pure soft-quarks in (3.70) and (3.72) rather than the loop momentum of the soft gluon (cf. Fig. 3.19). Upon exponentiation the double logarithm in (3.125) modifies (3.70) and (3.72) to

$$\begin{aligned} f_1(q_+, q_-) &= 1 + \frac{\alpha_s}{4\pi} \int_{q_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{q_+} \frac{dk_+}{k_+} e^{-s(q_+/k_+, p_{\eta-}/k_-)} \left[2C_F f_1(k_+, k_-) \right], \\ f_2(q_+, q_-) &= 1 + \frac{\alpha_s}{4\pi} \int_{q_-}^{p_{2-}} \frac{dk_-}{k_-} \int_{m_2^2/k_-}^{q_+} \frac{dk_+}{k_+} e^{-s(q_+/k_+, p_{\eta-}/k_-)} \\ &\quad \times \left[2C_F f_2(k_+, k_-) + \left(C_F - \frac{C_A}{2} \right) f_1(k_+, k_-) + \frac{C_A}{2} \right], \end{aligned} \quad (3.126)$$

where we introduced the notation

$$s(r_+, r_-) = \frac{\alpha_s C_F}{2\pi} \ln r_+ \ln r_-. \quad (3.127)$$

Taking also the global Sudakov factor generated by diagrams such as D'_9 in Fig. 3.9 into account, the final form for the double-logarithmic contributions to the form factor $F(\gamma)$ becomes

$$F(\gamma) \simeq \xi_0 \exp \left\{ -\frac{\alpha_s C_F}{4\pi} L^2 \right\} \left(2 \frac{1 + \bar{u}_0}{\bar{u}_0^3} f_2(m_2, m_2) + \frac{C_A}{2C_F \bar{u}_0^3} (1 - f_1(m_2, m_2)) - \frac{1}{\bar{u}_0^2} \right). \quad (3.128)$$

These two equations, (3.126) and (3.128), are the main result of our analysis. They encode the complicated interplay of soft-quark and soft-gluon effects to the soft-overlap form factor to all orders in the double-logarithmic approximation. Evaluated at tree-level, one-loop and two-loop, they match with the previously obtained results at fixed order in (3.2), (3.21) and (3.52). We note that their level of complexity is remarkable for the fact that they resum only double-logarithmic corrections.

Before we analyze the structure and the solution of these equations in more detail, we perform an independent check at fixed order in the next chapter.

So far we were extracting double logarithms by means of a diagrammatic method outlined in Sec. 3.1.1. In this chapter we will calculate the equivalent logarithms at fixed order by a different method. To this end, we discuss a factorization formula based on SCET. Endpoint divergences prevent us from resumming logarithms to all orders, but we will be able to determine the double logarithms at fixed order up to the 2-loop level, which serves as an independent check of the integral equations in (3.126). We will first give the factorization formula for general heavy-to-light form factors at large hadronic recoil. This will allow us to set up a pole-cancellation argument up to the 2-loop level. Finally, we will calculate the only missing ingredient to this argument to complete the independent check.

4.1 BARE FACTORIZATION FORMULA

In the introductory section about SCET (cf. Sec. 1.2) we discussed SCET-I. In SCET-I the dynamical modes are hard-collinear and soft with interactions among the two allowed by momentum conservation, and the hard-collinear virtuality being parametrically larger than the soft virtuality. For heavy-to-light form factors the energetic final state has the same virtuality as the soft dynamics in the heavy meson, because both are dominated by non-perturbative dynamics. Since, the virtuality of hard-collinear modes is larger than the soft virtuality, we need to introduce a second energetic mode that has the same virtuality as soft modes: the collinear mode (cf. Table 1.1). Because the interaction of a soft and collinear mode gives a hard-collinear mode, heavy-to-light form factors are described by a total of four different physical modes: Two perturbative modes, hard ($\mu_h^2 \sim m_b^2$) and hard-collinear ($\mu_{hc}^2 \sim m_b \Lambda_{\text{QCD}}$), and two non-perturbative modes, soft and collinear ($\mu_s^2 \sim \mu_c^2 \sim \Lambda_{\text{QCD}}^2$), where μ_i^2 denotes the typical virtuality of mode i . The scaling of the modes is given in Table 1.1 with hard scale $Q = m_b$ and power counting parameter $\lambda = \Lambda_{\text{QCD}}/m_b$. This implies that it requires a two-step matching procedure to arrive at a theory where only soft and collinear modes, i. e. the modes with the lowest virtuality, are dynamical. This theory is commonly referred to as SCET-II. In the following we describe the matching procedures necessary to incorporate the hard(-collinear) dynamics in the low scale theory SCET-II, discuss the required hadronic matrix elements that involve light-cone distribution amplitudes (LCDAs) and give the bare factorization formula of the soft-overlap form factor.

4.1.1 Matching corrections

In the the first matching step one integrates out the hard mode at the scale m_b and matches onto the intermediate theory SCET-I. For general $B \rightarrow P$ form factors, where P is a light pseudoscalar meson, the relevant operators in SCET-I are A -type and B -type operators [130]

$$J_A = \bar{\chi}_{hc} h_v, \quad J_B = \bar{\chi}_{hc} \mathcal{A}_{hc,\perp} h_v, \quad (4.1)$$

where χ_{hc} and $\mathcal{A}_{hc,\perp}^\mu$ are gauge-invariant building blocks defined in (1.61) and (1.62), respectively. The nomenclature of A -type and B -type operators reflects the fact that an B -type operator is power

suppressed with respect to an A -type operator due to an additional field that comes with a power-suppression. We note that both operators in (4.1) contribute at the same power, because the matching of J_A to SCET-II is power suppressed with respect to the matching of J_B . Schematically, heavy-to-light form factors at large hadronic recoil factorize for some current Γ_i according to

$$\langle P(p') | (\bar{q} \Gamma_i b)(0) | B(p) \rangle = F_i(q^2) + \Delta F_i(q^2) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_B}\right), \quad (4.2)$$

where $F_i(q^2)$ and $\Delta F_i(q^2)$ collect the matching coefficient and matrix elements of the A -type and B -type operators in (4.1), respectively, and $q^2 = (p - p')^2$. It turns out that the B -type operator in $\Delta F_i(q^2)$ can be matched to SCET-II operators with endpoint-finite convolution integrals [45], enabling a consistent extraction of large logarithms by means of RGE equations. For $F_i(q^2)$ we can write

$$F_i(q^2) = H_i(m_b, E) \xi(q^2), \quad (4.3)$$

with

$$\xi(q^2) = \frac{1}{2E} \langle P(p') | J_A(0) | B(v) \rangle, \quad (4.4)$$

where E is the energy of the pseudoscalar meson in the final state, $H_i(m_b, E)$ are the hard-matching coefficients and $\xi(q^2)$ is referred to as the soft-overlap term. We note that while H_i depends on the current Γ_i under consideration, $\xi(q^2)$ is independent of the current [79, 131], which is particularly important for phenomenology. We will see later that matching the time-ordered product

$$i \int d^4x T \{ J_A(0), \mathcal{L}_{\text{SCET-I}}(x) \}, \quad (4.5)$$

with subleading terms in the Lagrangian $\mathcal{L}_{\text{SCET-I}}$ onto SCET-II operators leads to endpoint-divergent convolution integrals. Since our aim is to further understand the structure of precisely these endpoint-divergent terms, we single out the terms that are related to the A -type operator to be able to concentrate only on endpoint-divergent structures. This is achieved by choosing the current

$$\Gamma_i \rightarrow \Gamma = \frac{\not{n} \not{\bar{n}}}{4}. \quad (4.6)$$

With the replacement (4.6) the form factor in (4.2) becomes

$$\langle P(p') | (\bar{q} \Gamma b)(0) | B(p) \rangle = H(m_b, E) \xi(q^2) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_B}\right), \quad (4.7)$$

and we see that there is no term $\Delta F_\Gamma(q^2)$ because its matching coefficient vanishes. The simplification in (4.7) is the reason why we defined the form factor $F(\gamma)$ for $B_c \rightarrow \eta_c$ as we did in (2.6).¹

Matching the time-ordered product of J_A with the SCET-I Lagrangian in (4.5) onto SCET-II the following operators are required for $B \rightarrow P$ transitions assuming massless quarks [132]

$$\begin{aligned} O_1 &= g_s^2 \left[\bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 \chi_c(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\not{n}}{4} \gamma_5 \mathcal{H}_v(0) \right], \\ O_2 &= g_s^2 \left[\bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 i \not{\phi}_\perp \chi_c(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right], \\ O_3 &= g_s^2 \left[\bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 \mathcal{A}_{c,\perp}(r\bar{n}) \chi_c(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right], \\ O_4 &= g_s^2 \left[\bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 \chi_c(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \mathcal{A}_{s,\perp}(\sigma n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right]. \end{aligned} \quad (4.8)$$

¹ Here the form factors have argument γ or q^2 . The translation between both cases is straightforward with the help of the definition of γ in (2.4).

In analogy to the definitions for hard-collinear fields in (1.61) and (1.62) the gauge-invariant building blocks in the collinear sector read

$$\chi_c = W_c^\dagger \xi_c, \quad \mathcal{A}_{c,\perp} = W_c^\dagger (iD_{c,\perp} W_c), \quad (4.9)$$

and in the soft sector,

$$Q_s = S_n^\dagger q_s, \quad \mathcal{H}_v = S_n^\dagger h_v, \quad \mathcal{A}_{s,\perp} = S_n^\dagger (iD_{s,\perp} S_n), \quad (4.10)$$

with soft Wilson lines S_n along the n^μ direction of the energetic final-state meson P . In (4.8) the collinear fields are non-local along the light-cone in the $\bar{n}$ direction as appropriate (cf. Sec. 1.2) and the soft fields are non-local in the n direction which follows from the Fourier transformation of the momentum-space matching coefficients, see below.

For the four-quark operators O_1 and O_2 the tree-level matching coefficients can be calculated by decomposing the Dirac structure of the diagrams in Fig. 3.1 with appropriate external momenta, and diagrams with an additional external gluon can be used to match onto the operators O_3 and O_4 [132]. For O_1 and O_2 the respective LO coefficients are given by

$$D_1^{(0)}(u, \omega) = -\frac{C_F}{N_c} \frac{1 + \bar{u}}{4E^2 \bar{u}^2 \omega}, \quad D_2^{(0)}(u, \omega) = -\frac{C_F}{N_c} \frac{1}{4E^2 u \bar{u}^2 \omega^2}, \quad (4.11)$$

where E is the energy of the energetic pseudoscalar meson P , ω is the light-cone momentum in the $\bar{n}^\mu$ direction of the soft spectator quark in the B meson, and u as well as $\bar{u} = 1 - u$ are the light-cone momentum fractions in the n^μ direction of the quark and anti-quark that form the collinear final-state meson P . Similarly, the matching coefficients for O_3 and O_4 are

$$D_3^{(0)}(\alpha_1, \alpha_2, \alpha_3, \omega) = \frac{1}{8E^2(\alpha_2 + \alpha_3)^2 \omega^2} \left[-4 \frac{C_F}{N_c} + \frac{\alpha_2 + \alpha_3}{\alpha_2} \frac{C_A}{N_c} - \frac{2C_F}{N_c} \frac{(\alpha_2 + \alpha_3)^2}{\alpha_2(\alpha_1 + \alpha_3)} \right],$$

$$D_4^{(0)}(u, \omega, \xi) = \frac{-1}{8E^2 \bar{u}^2 (\omega + \xi)^2} \left[\left(\frac{2C_F \bar{u} - C_A}{N_c} \right) \frac{\xi}{\omega} + \frac{2C_F - C_A}{N_c} \right], \quad (4.12)$$

where ξ denotes the soft light-cone momentum of the gluon, and the large collinear momentum is distributed for O_3 among the quark (α_1), the anti-quark (α_2), and the gluon (α_3), with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. In [132] the matching was performed for massless quarks. For our purposes, we require the corresponding matching calculation in the presence of massive quarks, which was performed in [44]. The result is that the matching coefficients for operators O_{1-4} remain unchanged, but an additional operator accounting for the mass in the numerator of the relevant diagrams has to be introduced

$$O_m = g_s^2 \left[\bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 \chi_c(s\bar{n}) \right] \left[\bar{Q}_s(tn) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) \right]. \quad (4.13)$$

The operators O_{1-4} all have a power-suppression in either the soft or the collinear sector. The operator O_m , on the other hand, is itself leading power, but its matching coefficient comes with a power-suppression

$$D_m^{(0)}(u, \omega) = \frac{C_F}{N_c} \frac{1}{4E^2 \bar{u} \omega^2} \left(m_1 \frac{\bar{u}}{u} + m_2 \frac{u}{\bar{u}} \right), \quad (4.14)$$

since it is proportional to the quark masses. We now turn to the matrix elements that originate from the operators $O_{1-4,m}$ in the context of $B \rightarrow P$ transitions.

4.1.2 Hadronic matrix elements

In SCET-II interactions among the soft and collinear sectors are power suppressed. This implies that states and Fock spaces decompose into products of soft and collinear states and Fock spaces. In $B \rightarrow P$ transitions at large recoil the B -meson is entirely described by soft dynamics after the heavy mass of the b quark is integrated out in the first matching step. The pseudoscalar meson P , on the other hand, is purely collinear. Consequently, the matrix element $\langle P(p') | \dots | B(p) \rangle$ collapses into products of vacuum-meson matrix elements in the soft and collinear sectors. In the following we will express soft and collinear matrix elements associated with the operators $\mathcal{O}_{1-4,m}$ in (4.8) and (4.13) in terms of the corresponding LCDAs and start with the soft sector, i. e. B -meson matrix elements.

The leading power two-particle B -meson LCDA $\phi_B^+(\omega)$ is defined as

$$\langle 0 | \bar{Q}_s(\tau n) \not{n} \gamma_5 \mathcal{H}_v(0) | B(v) \rangle = i f_B m_B \int_0^\infty d\omega e^{-i\omega\tau} \phi_B^+(\omega), \quad (4.15)$$

where f_B is the (scale-dependent) B -meson decay constant defined in HQET. Here and in the following we keep the dependence on the renormalization scale μ implicit, since for our application of interest it is currently unknown how to perform a consistent renormalization due to endpoint divergences. The subleading power two-particle LCDA $\phi_B^-(\omega)$ is defined via the other light-cone projection of the light-quark field,

$$\langle 0 | \bar{Q}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) | B(v) \rangle = -i f_B m_B \int_0^\infty d\omega e^{-i\omega\tau} \phi_B^-(\omega). \quad (4.16)$$

The last missing ingredient is the matrix element of the soft sector of $\mathcal{O}_4$ that defines the three-particle LCDA $\phi_{3B} = \Psi_A - \Psi_V$ (cf. [133])

$$\langle 0 | \bar{Q}_s(\tau n) \mathcal{A}_{s,\perp}(u\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_v(0) | B(v) \rangle = i f_B m_B \int_0^\infty d\omega \int_0^\infty \frac{d\xi}{\xi} e^{-i\omega\tau} e^{-i\xi(u\tau)} \phi_{3B}(\omega, \xi). \quad (4.17)$$

The three LCDAs $\phi_B^+(\omega)$, $\phi_B^-(\omega)$ and $\phi_{3B}(\omega, \xi)$ are related by an equation of motion [127]

$$\omega \phi_B^-(\omega) - m_2 \phi_B^+(\omega) + \int_0^\omega d\eta [\phi_B^+(\eta) - \phi_B^-(\eta)] = 2 \int_0^\omega d\eta \int_{\omega-\eta}^\infty \frac{d\xi}{\xi} \frac{\partial}{\partial \xi} \phi_{3B}(\eta, \xi), \quad (4.18)$$

which has a term that explicitly originates from the presence of a non-vanishing spectator-quark mass m_2 .

To be able to analyze which combinations of LCDAs and matching coefficients generate an endpoint divergence, which occur in the limit of small momenta, we have to understand the behavior of the LCDAs in the limit of small spectator-quark momenta. This can be done by inspecting their RG equations for large scales $\mu \rightarrow \infty$. The result is that for massless spectator quarks B -meson LCDAs are expected to scale as (see e. g. [134])

$$\phi_B^+(\omega; \mu) \sim \omega, \quad \phi_B^-(\omega; \mu) \sim \omega^0, \quad \phi_{3B}(\omega, \xi; \mu) \sim \omega \xi^2. \quad (4.19)$$

for small momenta. However, the scaling of $\phi_{3B}(\omega, \xi)$ has to be altered when taking masses into account. This can be seen by performing a derivative of (4.18) with respect to ω and setting $\omega = 0$

$$m_2 (\phi_B^+)'(0) = 2 \int_0^\infty \frac{d\xi}{\xi} \frac{\partial}{\partial \xi} \phi_{3B}(0, \xi), \quad (4.20)$$

which contradicts (4.19). Even repeating the steps that lead to (4.20) with the assumption that $\phi_B^+(\omega; \mu) \sim \omega^0$ gives a non-vanishing l.h.s., implying that $\phi_{3B}(\omega, \xi)$ must be finite as $\omega \rightarrow 0$. This leads to another set of endpoint-divergent moments of the LCDAs as we discuss below.

Coming to the collinear sector, the leading power LCDA of a light pseudo-scalar meson is defined as

$$\langle P(p') | \bar{\chi}_c(0) \not{n} \gamma_5 \chi_c(s\bar{n}) | 0 \rangle = -2i f_P E \int_0^1 du e^{2iEs\bar{u}} \phi(u). \quad (4.21)$$

The matrix element of the collinear part of $\mathcal{O}_3$ leads to a subleading power three-particle LCDA [135, 136]

$$\langle P(p') | \bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 \mathcal{A}_{c,\perp}(vs\bar{n}) \chi_c(s\bar{n}) | 0 \rangle = 2iE f_{3P} \int \frac{\mathcal{D}\alpha}{\alpha_3} e^{2iE(\alpha_2 s + \alpha_3 s v)} \phi_3(\alpha_1, \alpha_2, \alpha_3), \quad (4.22)$$

where we defined the integration measure

$$\int \mathcal{D}\alpha \equiv \int_0^1 d\alpha_2 \int_0^1 d\alpha_1 \int_0^1 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3). \quad (4.23)$$

The collinear part appearing in $\mathcal{O}_2$ can be written in terms of ϕ_3 , and the subleading power two-particle LCDA ϕ_σ associated with the pseudo-tensor current. In $d = 4$ dimensions, [44]

$$\begin{aligned} & \langle P(p') | \bar{\chi}_c(0) \frac{\not{n}}{2} \gamma_5 i \not{\phi}_\perp \chi_c(s\bar{n}) | 0 \rangle \\ &= 2iE \int_0^1 du e^{2iEs\bar{u}} \left\{ \frac{f_P \tilde{\mu}_P}{6} \phi_\sigma(u) - f_{3P} \int_0^u d\alpha_1 \int_{u-\alpha_1}^1 \frac{d\alpha_3}{\alpha_3^2} \phi_3(\alpha_1, 1 - \alpha_1 - \alpha_3, \alpha_3) \right\}, \end{aligned} \quad (4.24)$$

where $\tilde{\mu}_P = \mu_P - (m_1 + m_2)$, and $\mu_P = m_P^2/(m_1 + m_2)$ is the chirally-enhanced mass ratio. Similar to (4.18) the LCDAs for the light pseudoscalar meson can be related via equations of motion, which in this case involves an additional LCDA, the two-particle subleading power LCDA ϕ_p associated with the pseudo-scalar quark current. The relations read [136]

$$\begin{aligned} & \frac{f_P \tilde{\mu}_P}{6} \phi'_\sigma(u) + (2u - 1) f_{P\mu P} \phi_p(u) - (m_1 - m_2) f_P \phi(u) \\ &= -2f_{3P} \int \frac{\mathcal{D}\alpha}{\alpha_3} \phi_3(\{\alpha_i\}) (\delta(\alpha_1 - u) - \delta(\alpha_2 - \bar{u})), \end{aligned} \quad (4.25)$$

and

$$\begin{aligned} & \frac{f_P \tilde{\mu}_P}{6} \left((2u - 1) \phi'_\sigma(u) - 4\phi_\sigma(u) \right) + f_{P\mu P} \phi_p(u) - (m_1 + m_2) f_P \phi(u) \\ &= 2f_{3P} \int \frac{\mathcal{D}\alpha}{\alpha_3} \phi_3(\{\alpha_i\}) \left(\delta(\alpha_1 - u) + \delta(\alpha_2 - \bar{u}) + \frac{2}{\alpha_3} \left[\theta(\alpha_1 - u) - \theta(\bar{u} - \alpha_2) \right] \right), \end{aligned} \quad (4.26)$$

An analysis of the RG equations can be used to determine the expected scaling of the LCDAs for small momenta. Under the assumption of vanishing quark masses the expectation is [136]

$$\phi(u; \mu) \sim \phi_\sigma(u; \mu) \sim 6u\bar{u}, \quad \phi_p(u; \mu) \sim 1, \quad \phi_3(\{\alpha_i\}; \mu) \sim \frac{360 f_{3P}}{f_P \mu_P} \alpha_1 \alpha_2 \alpha_3^2. \quad (4.27)$$

Again this expectation has to be modified in the presence of masses. In this case, the result of an analysis similar to the one around (4.20) is that ϕ_3 remains finite for $\alpha_2 \rightarrow 0$ [44].

4.1.3 Bare factorization formula

In this section we combine the hard-collinear matching coefficients with the hadronic matrix elements and discuss the occurring endpoint divergences as well as their consequences. To combine the position-space matrix elements given in the previous section with the momentum-space matching coefficients in Sec. 4.1.1 we transform the latter into position space. For the matching coefficients corresponding to two-particle Fock-states the relation reads

$$D_i(\omega, \bar{u}; E) = \int ds e^{i2Es\bar{u}} \int dt e^{-i\omega\tau} \tilde{D}_i(s, \tau; E), \quad \text{for } i \in \{1, 2, m\} \quad (4.28)$$

and similarly for the coefficients corresponding to three-particle Fock-states

$$\begin{aligned} D_3(\omega, \alpha_2, \alpha_3; E) &= \int dr ds e^{i2E(s\alpha_2+r\alpha_3)} \int d\tau e^{-i\omega\tau} \tilde{D}_3(r, s, \tau; E), \\ D_4(\omega, \xi, \bar{u}; E) &= \int ds e^{i2Es\bar{u}} \int d\tau d\sigma e^{-i(\omega\tau+\xi\sigma)} \tilde{D}_4(s, \tau, \sigma; E). \end{aligned} \quad (4.29)$$

Given these relations the soft-overlap contribution is given by the convolution of the position-space matching coefficients and matrix elements

$$\begin{aligned} 2E \xi(q^2) &= \sum_{i=1,2,m} \int ds d\tau \tilde{D}_i(s, \tau; E) \langle P(p') | O_i(s, \tau) | B(v) \rangle \\ &\quad + \int dr ds d\tau \tilde{D}_3(r, s, \tau; E) \langle P(p') | O_3(r, s, \tau) | B(v) \rangle \\ &\quad + \int ds d\tau d\sigma \tilde{D}_4(s, \tau, \sigma; E) \langle P(p') | O_4(s, \tau, \sigma) | B(v) \rangle, \end{aligned} \quad (4.30)$$

where the factor of $2E$ originates from the definition of the soft-overlap term in (4.4). Inserting the definitions of the LCDAs and tree-level matching coefficients discussed in previous sections leads to a factorization formula. Given that it is possible to rearrange terms by means of the equation-of-motion identities among the LCDAs the representation of the factorization formula is not unique. We find it most convenient to express it in the form [44]

$$\begin{aligned} \xi^{\text{LO}}(q^2) &= \frac{\alpha_s C_F}{4\pi} \frac{\pi^2 f_B f_P m_B}{N_c E^2} \left\{ \right. \\ &\quad \int_0^\infty d\omega \int_0^1 du \left[\frac{\phi_B^-(\omega) + \phi_B^+(\omega)}{\omega} \frac{1+\bar{u}}{\bar{u}^2} \phi(u) - 2 \frac{\phi_B^+(\omega)}{\omega} \frac{\phi(u)}{\bar{u}} \right. \\ &\quad \left. + \frac{\phi_B^+(\omega)}{\omega^2} \left(-m_1 \frac{\phi(u)}{\bar{u}} - 2m_2 \frac{\phi(u)}{\bar{u}^2} + 3\mu_P \frac{\phi_P(u)}{\bar{u}} + \frac{\tilde{\mu}_P}{6} \frac{\phi'_\sigma(u)}{\bar{u}} \right) \right] \\ &\quad - \frac{C_A - 2C_F}{C_F} \frac{f_{3P}}{f_P} \int_0^\infty d\omega \frac{\phi_B^+(\omega)}{\omega^2} \int \mathcal{D}\alpha \frac{\phi_3(\{\alpha_i\})}{\alpha_2 \alpha_3 (\alpha_2 + \alpha_3)} \\ &\quad \left. + \frac{C_A - 2C_F}{C_F} \int_0^\infty d\omega \int_0^\infty d\xi \frac{\phi_{3B}(\omega, \xi)}{\omega \xi (\omega + \xi)} \int_0^1 du \frac{\phi(u)}{\bar{u}^2} \right\}, \end{aligned} \quad (4.32)$$

where the superscript LO refers to the fact that only tree-level matching is incorporated. In this form all three-particle LCDAs are suppressed by $\sim 1/N_c^2$ since $C_A - 2C_F = 1/N_c$. Combining the

asymptotic limits of the LCDAs with the explicit form of (4.31), it is straightforward to read off which terms are endpoint finite or divergent. Defining the schematic notation

$$\langle x^\alpha \rangle_f = \int dx x^\alpha f(x), \quad (4.33)$$

for moments of LCDAs, we identify that the moments

$$\left\langle \frac{1}{\omega^2} \right\rangle_{\phi_B^+}, \left\langle \frac{1}{\omega} \right\rangle_{\phi_B^-}, \left\langle \frac{1}{\omega \xi (\omega + \xi)} \right\rangle_{\phi_{3B}^+}, \left\langle \frac{1}{\bar{u}^2} \right\rangle_{\phi}, \left\langle \frac{1}{\bar{u}} \right\rangle_{\phi_p}, \left\langle \frac{1}{\alpha_2 \alpha_3 (\alpha_2 + \alpha_3)} \right\rangle_{\phi_3}, \quad (4.34)$$

in (4.31) are divergent at the endpoint given the asymptotic scaling of LCDAs in (4.19), (4.27) and the discussions thereafter. In contrast to UV divergences that can be removed by renormalization at the level of e. g. matching coefficients or LCDAs, these divergences originate from the interplay of matching coefficients and LCDAs. One strategy to deal with them is to systematically subtract the limits at the endpoints, an idea similar to that of subtraction schemes (cf. Sec. 1.5) in the context of IR divergences. This strategy employs so-called refactorization identities at the endpoints and was first discussed in [44] and successfully implemented in [35]. To this date, it is sufficient in all successful implementations to employ additive refactorization identities [34–43], which means that the refactorization identities themselves do not introduce new endpoint divergences. However, in more complicated situations—e. g. exclusive hard processes, which are the focus of this work—this very complication arises [44], making the removal of endpoint divergences significantly more challenging.

While endpoint divergences prevent the resummation of large logarithms by means of RG equations, it is possible to extract logarithms at fixed order in a perturbative setting provided all endpoint divergences are regulated properly. Using the factorization formula (4.31) for bare (unrenormalized) quantities to extract logarithms at fixed order within the non-relativistic setup for $B_c \rightarrow \eta_c$ transitions will be the objective of the next section.

4.2 DOUBLE LOGARITHMS IN $B_c \rightarrow \eta_c$ FORM FACTORS

In the previous section we presented a SCET-based factorization formula for the soft-overlap contribution $\xi(q^2)$ to heavy-to-light form factors at large recoil (cf. (4.31)). Whereas we kept the notation generic in the previous section we now come back to the specific case of $B_c \rightarrow \eta_c$ transitions in the non-relativistic limit, which can be treated perturbatively and that was the focus of the diagrammatic analysis in chapter 3. We will now calculate double-logarithmic contributions to the associated form factor $F(\gamma)$ using exclusively the factorization formula for $\xi(q^2)$. In this setup the LCDAs are perturbative and at LO they simply project onto their on-shell values. For the two-particle B -meson LCDAs this amounts to a delta function for the light-cone momentum fraction ω

$$\phi_B^{+, (0)}(\omega) = \phi_B^{-, (0)}(\omega) = \delta(\omega - m_2). \quad (4.35)$$

Similarly, the two-particle LCDAs of the light meson $\phi(u)$ and $\phi_p(u)$ project the collinear momentum fraction u to the mass ratio u_0

$$\phi^{(0)}(u) = \phi_p^{(0)}(u) = \delta(u - u_0). \quad (4.36)$$

The three-particle LCDAs for the B -meson $\phi_{3B}(\omega, \xi)$ and the light meson $\phi_3(\{\alpha_i\})$ as well as $\tilde{\mu}_P$ are suppressed by one power of α_s and therefore do not contribute at tree-level. This implies that $\mathcal{O}_{2-4}$

are not required to describe $F(\gamma)$ at LO and only the operators $\mathcal{O}_1$ and $\mathcal{O}_m$ contribute at this level. Using $\mu_P = m_\eta = m_1 + m_2$ and $H(m_b, E_\eta) = 1 + \mathcal{O}(\alpha_s)$, we recover the tree-level result

$$F^{(0)}(\gamma) = \xi_0 \frac{2 + \bar{u}_0}{\bar{u}_0^3}, \quad (4.37)$$

which agrees with (3.2).

Beyond tree-level the LCDAs start to have broader support, which eventually leads to endpoint divergences as discussed in the previous section. We therefore have to introduce suitable regulators and we will see that some of the divergences—so-called rapidity divergences—are not regulated in $d \neq 4$ dimensions, which requires to implement an additional regulator besides ε . The need to introduce such rapidity regulators is by now a well-known feature of SCET-II [137, 138] and we discuss them further below. In the following we extend the analysis of the factorization formula (4.31) beyond tree-level, which will serve as an important check of the integral equations (3.126) and their implementation in the form factor $F(\gamma)$.

4.2.1 Next-to-leading order

At higher perturbative orders all regions, i. e. hard, hard-collinear, collinear and soft regions, receive distinct corrections in α_s . Corrections in the hard and hard-collinear regions manifest as corrections to the matching coefficients, while those in the collinear and soft regions appear in the respective LCDAs. We work in the double-logarithmic approximation. At this level, it is sufficient to consider double poles in ε and an analytic rapidity regulator, which we have yet to define, combined with the respective virtualities and rapidities.

MATCHING CORRECTIONS: The matching coefficient $H(m_b, E_\eta)$ in (4.7) originates from radiative corrections to the weak-interaction vertex at virtualities of $\mathcal{O}(m_b^2)$. It is well known that the leading singularities of the bare coefficient exponentiate to all orders

$$H(m_b, E_\eta) \simeq \exp \left\{ -\frac{\alpha_s C_F}{4\pi} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{4E_\eta^2} \right)^\varepsilon \right\} = 1 - \frac{\alpha_s C_F}{4\pi} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{4E_\eta^2} \right)^\varepsilon + \mathcal{O}(\alpha_s^2). \quad (4.38)$$

Combining this with the soft-overlap term at LO gives

$$F_h^{(1)}(\gamma) \simeq -\frac{C_F}{\varepsilon^2} \left(\frac{\mu^2}{4E_\eta^2} \right)^\varepsilon F^{(0)}(\gamma). \quad (4.39)$$

The ε -expansion of $H(m_b, E_\eta)$ generates logarithms in the ratio of the factorization scale μ and the intrinsic hard scale of the process $\mu_h^2 \sim m_b^2 \sim E_\eta^2$.

The matching corrections at the hard-collinear scale are slightly more involved. Since contributions from the operators $\mathcal{O}_{2-4}$ start only at the one-loop level, they do not contribute at the double-logarithmic level and there is no need to consider their matching corrections. The perturbative corrections of the remaining operators do contribute at the double-logarithmic level and their required double poles in ε were calculated in [44]. For D_1 they are given by

$$D_1^{(1)}(u, \omega) \simeq -\frac{C_F}{N_c} \frac{1 + \bar{u}}{4E^2 \bar{u}^2 \omega} \frac{4C_F}{\varepsilon^2} \left(\frac{\mu^2}{2\omega E_\eta} \right)^\varepsilon, \quad (4.40)$$

and for D_m by

$$D_m^{(1)}(u, \omega) \simeq \frac{C_F}{N_c} \frac{1}{4E^2 \bar{u} \omega^2} \frac{4}{\varepsilon^2} \left(\frac{\mu^2}{2\omega E_\eta} \right)^\varepsilon \left\{ \left(m_1 \frac{\bar{u}}{u} + m_2 \frac{3}{2\bar{u}} \right) C_F - m_2 \frac{C_A}{4\bar{u}} \right\}. \quad (4.41)$$

We note that the dependence on ω is fixed on dimensional grounds, while the same constraint does not apply for the dimensionless variable u . Combining (4.40) and (4.41) with the tree-level expression of the respective LCDAs gives

$$F_{hc}^{(1)}(\gamma) \simeq \frac{2C_F}{\varepsilon^2} \left(\frac{\mu^2}{2m_\eta E_\eta} \right)^\varepsilon F^{(0)}(\gamma) + \frac{\xi_0}{\varepsilon^2} \left(\frac{\mu^2}{2m_\eta E_\eta} \right)^\varepsilon \left(6C_F \frac{1 + \bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{\bar{u}_0^3} \right), \quad (4.42)$$

where the virtuality depends on the hard-collinear scale $\mu_{hc}^2 \sim m_\eta E_\eta$ as appropriate and an ε -expansion generates the corresponding logarithms. In the virtuality we neglected terms that are $O(1)$ according to our power counting, which do not contribute at the double-logarithmic level. The singularities in the hard-collinear region clearly do not exponentiate since they modify both the color structure and the dependence on the mass ratios with respect to the LO result in (3.2).

HADRONIC MATRIX ELEMENTS: The double poles stemming from the matching coefficients must be canceled by those originating from the matrix elements. Here, there are two distinct sources of double poles: First, the LCDAs themselves develop double poles. Second, as the LCDAs start to have a non-trivial support over the entire integration range, at NLO and beyond, there may be double poles generated by endpoint divergences.

We begin by examining the collinear sector. The corresponding LCDAs are single-logarithmic quantities because they follow Brodsky-Lepage-like RG equations [139–141]. Thus, the only source that can possibly give double poles in this sector are endpoint-divergent convolutions. In the form of (4.31) there are endpoint-divergent moments involving the three-particle LCDA ϕ_3 . Multiplying the difference of the equations-of-motions relations in (4.25) and (4.26) with $\bar{u}^{-2}$, we derive

$$2f_{3P} \int \mathcal{D}\alpha \frac{\phi_3(\{\alpha_i\})}{\alpha_2 \alpha_3 (\alpha_2 + \alpha_3)} = f_P \int_0^1 du \left(m_2 \frac{\phi(u)}{\bar{u}^2} - \mu_P \frac{\phi_P(u)}{\bar{u}} - \frac{\tilde{\mu}_P}{6} \frac{\phi'_\sigma(u)}{\bar{u}} \right) + 2f_{3P} \int \mathcal{D}\alpha \frac{\phi_3(\{\alpha_i\})}{\alpha_3 (\alpha_2 + \alpha_3)^2}. \quad (4.43)$$

On the r.h.s. moments of ϕ_3 are endpoint-finite and we can hence express all endpoint-divergent moments entirely in terms two-particle LCDAs. Using this identity allows us to make an all-order statement about the contributions from the collinear sector to the double-logarithmic approximation of the form factor $F(\gamma)$ by collecting all endpoint-divergent convolutions of collinear LCDAs and setting all B -meson LCDAs to their tree-level values in (4.31). This yields

$$\bar{u}_0^2 \frac{F(\gamma)|_{\text{coll}}}{\xi_0} \simeq -\frac{u_0}{\bar{u}_0} + \bar{u}_0 \frac{2C_F - C_A}{2C_F} \int_0^1 \frac{du}{\bar{u}^2} \phi(u) + \frac{2\mu_P}{m_\eta} \int_0^1 \frac{du}{\bar{u}} \phi_P(u) + \frac{C_A}{2C_F} \int_0^1 \frac{du}{\bar{u}} \phi_{p+\sigma'}(u), \quad (4.44)$$

where the first term is an endpoint-finite moment of ϕ that only contributes at LO in the double-logarithmic approximation. All required collinear LCDAs have been computed in [127] at NLO. Since they originate from a single convolution, we expect that the endpoint divergences give one more pole in either the dimensional regulator ε or the additional regulator that we mentioned before. To arrive at a double pole it is therefore sufficient to focus on the UV-divergent pieces that

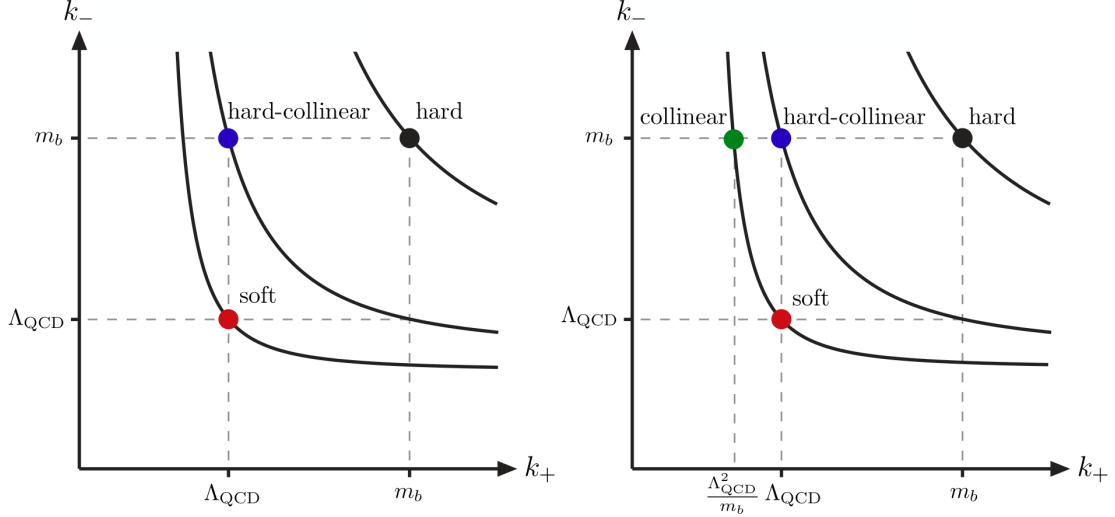


Figure 4.1: Modes in SCET-I (left panel) are on separate invariant-mass hyperbolas, whereas in SCET-II (right panel) the collinear and soft mode lie on the same hyperbola. The figures are taken from [44].

already come with a single ε -pole. Expanding the poles of the respective LCDAs in the limit $u \rightarrow 1$, where we expect the endpoint singularity to arise, gives

$$\begin{aligned}\phi^{(1)}(u \rightarrow 1) &\simeq \frac{2C_F}{\varepsilon} \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon \frac{1 + \bar{u}_0}{\bar{u}_0^2} \bar{u}, \\ \phi_p^{(1)}(u \rightarrow 1) &\simeq \frac{2C_F}{\varepsilon} \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon \frac{1 + \bar{u}_0}{\bar{u}_0}, \\ \phi_{p+\sigma'}^{(1)}(u \rightarrow 1) &\simeq \frac{2C_F}{\varepsilon} \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon,\end{aligned}\tag{4.45}$$

which shows that the integrals in (4.44) are ill-defined even in d space-time dimensions. The underlying problem is that dimensional regularization fails to distinguish between modes that lie in the (k_+, k_-) plane on the same hyperbola of invariant mass $k^2 = k_+ k_- + k_\perp^2$. In this case the ratio k_-/k_+ helps to distinguish between different regions on the same hyperbola, which is why they are referred to as rapidity divergences. Since in SCET-II there is an hyperbola of invariant mass that is occupied by more than one region it suffers from rapidity divergences in contrast to SCET-I where every hyperbola of invariant mass has at most one region (see Fig. 4.1). In case a rapidity divergence is not regularized by dimensional regularization—as is the case in the convolution integrals under consideration—a regulator that is sensitive to the ratio k_-/k_+ has to be introduced. We choose to implement an analytic regulator that modifies each integration measure according to [138]

$$d^d k \rightarrow d^d k \left(\frac{\nu}{k_-} \right)^\alpha,\tag{4.46}$$

where k^μ is the momentum of the spectator quark that flows out of the hard-collinear interaction vertex, i.e. $k_- = 2\bar{u}E_\eta$. At NLO, all endpoint-divergent convolutions in the collinear sector become

$$\int_0^1 du \frac{1}{\bar{u}} \left(\frac{\nu}{2\bar{u}E_\eta} \right)^\alpha = -\frac{1}{\alpha} \left(\frac{\nu}{2E_\eta} \right)^\alpha,\tag{4.47}$$

and are regularized by α . Inserting this into the endpoint-divergent moments we obtain

$$\begin{aligned} \int_0^1 \frac{du}{\bar{u}^2} \phi^{(1),\text{reg}}(u) &\simeq \frac{-2C_F}{\alpha\epsilon} \left(\frac{\mu^2}{m_\eta^2} \right)^\epsilon \left(\frac{v}{2E_\eta} \right)^\alpha \frac{1 + \bar{u}_0}{\bar{u}_0^2}, \\ \int_0^1 \frac{du}{\bar{u}} \phi_p^{(1),\text{reg}}(u) &\simeq \frac{-2C_F}{\alpha\epsilon} \left(\frac{\mu^2}{m_\eta^2} \right)^\epsilon \left(\frac{v}{2E_\eta} \right)^\alpha \frac{1 + \bar{u}_0}{\bar{u}_0}, \\ \int_0^1 \frac{du}{\bar{u}} \phi_{p+\sigma'}^{(1),\text{reg}}(u) &\simeq \frac{-2C_F}{\alpha\epsilon} \left(\frac{\mu^2}{m_\eta^2} \right)^\epsilon \left(\frac{v}{2E_\eta} \right)^\alpha. \end{aligned} \quad (4.48)$$

Inserting these moments into (4.44) the regulated NLO result in the collinear sector is given by

$$F_c^{(1)}(\gamma) \simeq \frac{-\xi_0}{\alpha\epsilon} \left(\frac{\mu^2}{m_\eta^2} \right)^\epsilon \left(\frac{v}{2E_\eta} \right)^\alpha \left(6C_F \frac{1 + \bar{u}_0}{\bar{u}_0^3} - C_A \frac{1}{\bar{u}_0^3} \right). \quad (4.49)$$

We note that the rapidity scale inherits its scaling from the definition in (4.46) and is large ($v \sim 2E_\eta$), while the virtuality in the collinear sector is small ($\mu^2 \sim m_\eta^2$). Expanding in ϵ and α gives the corresponding logarithms. In (4.49) the result of the expansion is not sensitive to the order in which the limits $\epsilon \rightarrow 0$ and $\alpha \rightarrow 0$ are taken. In general,—and we encounter such a case below—it is possible that the order of the two limits does not commute. In such a case it is important to remember that the regulator is introduced at the level of a loop integrand *before* we expand in the power counting of regions. To be consistent with the hard and hard-collinear region, where we effectively expand in α before we perform the loop integrations, it is mandatory that we expand in α before we expand in ϵ .

The analysis in the soft sector is more involved for two reasons. First, the B -meson LCDAs are themselves double-logarithmic quantities in contrast to the LCDAs in the collinear sector, because they obey Lange-Neubert-like evolution equations [142]. This can be understood by analyzing their Wilson lines. While LCDAs in the collinear sector only contain collinear Wilson lines in the $\bar{n}$ direction, the B -meson LCDAs contain time-like soft Wilson lines in the direction v of the heavy b quark originating on a technical level from the decoupling transformation of the HQET field h_v , and soft Wilson lines in the n direction to ensure gauge invariance. The cusp between both types of Wilson lines generates double logarithms, or equivalently double poles in ϵ . The second reason is that the argument ω of the B -meson LCDAs corresponds to the k_+ components of the spectator-quark momentum, but the regulator in (4.46) is defined with a k_- component, which makes a consistent implementation of the rapidity regulator more complicated. Proceeding in a similar fashion as in the collinear sector we write the three-particle LCDAs in terms of two-particle LCDAs. Employing the identity

$$\frac{1}{\omega\xi(\omega+\xi)} = \frac{1}{\omega(\omega+\xi)^2} + \frac{1}{\xi(\omega+\xi)^2}. \quad (4.50)$$

we notice that a moment of the three-particle LCDA ϕ_{3B} with the last term on the r.h.s. is endpoint finite and α_s suppressed and thus does not contribute at the double-logarithmic level. This allows us to use [44]

$$2 \int_0^\infty d\omega \int_0^\infty d\xi \frac{\phi_{3B}(\omega, \xi)}{\omega(\omega+\xi)^2} = \int_0^\infty \frac{d\omega}{\omega} \phi_B^+(\omega) \left(1 - \frac{m_2}{\omega} \right), \quad (4.51)$$

which can be derived from the equations of motion (4.18), to express ϕ_{3B} in terms of the leading power two-particle LCDA ϕ_B^+ . Setting collinear LCDAs to their LO expression and collecting all pieces in the soft sector that contribute at the double-logarithmic level we obtain

$$\frac{F(\gamma)|_{\text{soft}}}{\xi_0} \simeq \int_0^\infty d\omega \frac{m_2}{\omega} \left\{ \frac{1+\bar{u}_0}{\bar{u}_0^3} \left(\phi_B^-(\omega) + \frac{m_2}{\omega} \phi_B^+(\omega) \right) - \frac{\phi_B^+(\omega)}{\bar{u}_0^2} + \frac{C_A}{2C_F \bar{u}_0^3} \phi_B^+(\omega) \left(1 - \frac{m_2}{\omega} \right) \right\}. \quad (4.52)$$

As we mentioned previously it is more involved in the soft sector to implement the regulator (4.46). In addition, the result becomes sensitive to the order of expanding in α and ε through terms of the form

$$\frac{\Gamma(\alpha + \varepsilon)}{\alpha} \simeq \frac{1}{\alpha \varepsilon} - \frac{1}{\varepsilon^2}, \quad (4.53)$$

which need to be expanded first in α and then in ε in accordance with our discussion above. Thus, this rapidity divergence generates both mixed poles in α and ε and double poles in ε . The endpoint-divergent moments were calculated in [44] and are given by

$$\begin{aligned} \int_0^\infty d\omega \frac{m_2^2}{\omega^2} \phi_{B,\text{reg}}^{+, (1)}(\omega) &\simeq 2C_F \left(\frac{1}{\alpha \varepsilon} - \frac{1}{\varepsilon^2} \right) \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon \left(\frac{\nu}{m_2} \right)^\alpha - \frac{C_F}{\varepsilon^2} \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon, \\ \int_0^\infty d\omega \frac{m_2}{\omega} \phi_{B,\text{reg}}^{-, (1)}(\omega) &\simeq 4C_F \left(\frac{1}{\alpha \varepsilon} - \frac{1}{\varepsilon^2} \right) \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon \left(\frac{\nu}{m_2} \right)^\alpha - \frac{C_F}{\varepsilon^2} \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon. \end{aligned} \quad (4.54)$$

We note that in contrast to the rapidity scale in the collinear sector, here, both the rapidity scale and the virtuality are small, i. e. $\nu \sim m_2$ and $\mu^2 \sim m_2^2$. In (4.52) there is also an endpoint-finite moment of ϕ_B^+ that develops a double pole in ε due to the double-logarithmic nature of the B -meson LCDAs

$$\int_0^\infty d\omega \frac{m_2}{\omega} \phi_B^{+, (1)}(\omega) \simeq -\frac{C_F}{\varepsilon^2} \left(\frac{\mu^2}{m_2^2} \right)^\varepsilon. \quad (4.55)$$

Collecting all double poles in (4.52) gives

$$F_s^{(1)}(\gamma) \simeq \xi_0 \left(\frac{\mu^2}{m_\eta^2} \right)^\varepsilon \left(\frac{\nu}{m_\eta} \right)^\alpha \left(6C_F \frac{1+\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{\bar{u}_0^3} \right) \left(\frac{1}{\alpha \varepsilon} - \frac{1}{\varepsilon^2} \right) - \frac{C_F}{\varepsilon^2} \left(\frac{\mu^2}{m_\eta^2} \right)^\varepsilon F^{(0)}(\gamma), \quad (4.56)$$

Combining the results from all four regions in (4.39), (4.42), (4.49), and (4.56) we obtain the double-logarithmic contribution at NLO

$$F^{(1)}(\gamma) \simeq \xi_0 \left(6C_F \frac{1+\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{\bar{u}_0^3} \right) \frac{L^2}{2} - C_F L^2 F^{(0)}(\gamma) = \xi_0 L^2 \left(C_F \frac{1+2\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{2\bar{u}_0^3} \right) \quad (4.57)$$

where all poles in α and ε cancel and which agrees with (3.21). By inspecting (4.57) and the double poles that lead to it, we can identify two distinct source of double-logarithmic contributions. The logarithms proportional to the tree-level form factor arise from cancellations among the hard, hard-collinear and soft mode, and are connected to the soft-gluon configurations we discussed e. g. in Sec. 3.3. The remaining logarithms result from cancellations among the rapidity poles in the collinear and soft region. For instance, in the collinear sector these divergences can be traced back to the limit $u \rightarrow 1$, i. e. there is a substantial contribution from the collinear spectator quark becoming soft, which is why this contribution is coined the soft-overlap term. These contributions are related to the soft-quark configuration and their strong-rapidity ordering.

4.2.2 Next-to-next-to-leading order

One of the most important features of the integral equations (3.126) is their iterative nature. But this feature only becomes accessible at NNLO and beyond. Likewise, another important part of the integral equations is the exponential Sudakov factor in the integrand of the integral equations. Similarly to the iterative nature, this effect first comes into play at the two-loop order. It is therefore crucial to extend the independent check up to the two-loop level.

In the previous section we used the factorization formula (4.31) to systematically calculate all leading poles in combination with their corresponding virtualities and rapidity scales to determine all double-logarithmic corrections to the soft-overlap form factor. This strategy becomes significantly more difficult with increasing loop order. For this reason we will turn the argument around at the two-loop level. Instead of using pole cancellation as a cross-check of our calculation, we will make an ansatz for the coefficients of all possible poles with appropriate virtualities and rapidity scales, and use pole cancellation as a constraint to minimize the number of unknown coefficients that enter the final result at a given loop order.² To explain this strategy in more detail we repeat the NLO calculation employing this ansatz before we start the analysis at NNLO.

At NLO we can split the form factor into distinct contributions from the hard, hard-collinear, collinear and soft regions

$$F^{(1)}(\gamma) = F_h^{(1)}(\gamma) + F_j^{(1)}(\gamma) + F_c^{(1)}(\gamma) + F_s^{(1)}(\gamma), \quad (4.58)$$

where indices correspond to the nomenclature of the regions, except for the hard-collinear contributions which we abbreviate with j in this section for notational convenience. Following the discussion of the previous section we assign the appropriate virtualities and rapidity scales to the corresponding coefficients

$$\begin{aligned} F_h^{(1)}(\gamma) &\simeq \frac{h_{20}}{\varepsilon^2} \left(\frac{\mu^2}{4E_\eta^2} \right)^\varepsilon F^{(0)}(\gamma), \\ F_j^{(1)}(\gamma) &\simeq \frac{j_{20}}{\varepsilon^2} \left(\frac{\mu^2}{2m_\eta E_\eta} \right)^\varepsilon, \\ F_c^{(1)}(\gamma) &\simeq \left\{ \frac{c_{20}}{\varepsilon^2} + \frac{c_{11}}{\alpha\varepsilon} \right\} \left(\frac{\mu^2}{m_\eta^2} \right)^\varepsilon \left(\frac{v}{2E_\eta} \right)^\alpha, \\ F_s^{(1)}(\gamma) &\simeq \left\{ \frac{s_{20}}{\varepsilon^2} + \frac{s_{11}}{\alpha\varepsilon} \right\} \left(\frac{\mu^2}{m_\eta^2} \right)^\varepsilon \left(\frac{v}{m_\eta} \right)^\alpha. \end{aligned} \quad (4.59)$$

In (4.59) we defined a notation that relates a coefficient to its pole structure. For instance, for the collinear region a given coefficient c_{mn} has the pole structure

$$\frac{c_{mn}}{\varepsilon^m \alpha^n}. \quad (4.60)$$

In total we defined six coefficients in (4.59). Expanding virtualities and rapidity in ε and α there are poles in ε^{-2} , ε^{-1} , $\varepsilon^{-1}\alpha^{-1}$, α^{-1} , which have to vanish separately. In total this gives four constraints

² This strategy is based on the assumption that there are no new momentum regions starting to contribute at higher loop orders. Recently, there have been discussions about additional regions (see [65] for a review), but we expect that these regions only contribute at subleading power if they contribute at all.

which leaves two coefficients undetermined by pole cancellation. We have a certain freedom to choose these coefficients. Choosing the coefficients h_{20} and hc_{20} or, alternatively, h_{20} and c_{11} we arrive at the result

$$F^{(1)}(\gamma) \simeq \left(2h_{20}F^{(0)}(\gamma) + \frac{j_{20}}{2} \right) L^2 = \left(h_{20}F^{(0)}(\gamma) - \frac{c_{11}}{2} \right) L^2. \quad (4.61)$$

Implementing the coefficients

$$h_{20} = -C_F, \quad j_{20} = 2C_F F^{(0)}(\gamma) + \xi_0 \left(6C_F \frac{1+\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{\bar{u}_0^3} \right), \quad c_{11} = -\xi_0 \left(6C_F \frac{1+\bar{u}_0}{\bar{u}_0^3} - \frac{C_A}{\bar{u}_0^3} \right), \quad (4.62)$$

which can be read off from (4.39), (4.42), and (4.49) in (4.61) it is straightforward to see that we recover the well-known result at NLO in (3.21) in both cases.

At NNLO there are pure and mixed contributions. The pure contributions can be parameterized in analogy to the regions at one-loop with appropriate changes in the powers of the virtuality and rapidity scales

$$\begin{aligned} F_{hh}^{(2)}(\gamma) &\simeq \frac{h_{40}}{\varepsilon^4} \left(\frac{\mu^2}{4E_\eta^2} \right)^{2\varepsilon} F^{(0)}(\gamma), \\ F_{jj}^{(2)}(\gamma) &\simeq \frac{j_{40}}{\varepsilon^4} \left(\frac{\mu^2}{2m_\eta E_\eta} \right)^{2\varepsilon}, \\ F_{cc}^{(2)}(\gamma) &\simeq \left\{ \frac{c_{40}}{\varepsilon^4} + \frac{c_{31}}{\alpha \varepsilon^3} + \frac{c_{22}}{\alpha^2 \varepsilon^2} \right\} \left(\frac{\mu^2}{m_\eta^2} \right)^{2\varepsilon} \left(\frac{v}{2E_\eta} \right)^{2\alpha}, \\ F_{ss}^{(2)}(\gamma) &\simeq \left\{ \frac{s_{40}}{\varepsilon^4} + \frac{s_{31}}{\alpha \varepsilon^3} + \frac{s_{22}}{\alpha^2 \varepsilon^2} \right\} \left(\frac{\mu^2}{m_\eta^2} \right)^{2\varepsilon} \left(\frac{v}{m_\eta} \right)^{2\alpha}, \end{aligned} \quad (4.63)$$

where we accommodate our expectation that with each additional perturbative order there can potentially be one more power in the rapidity regulator. For the mixed terms we assume that the hard region factorizes completely and consequently mixed terms with the hard region are simple products of one-loop terms without genuine two-loop coefficients

$$F_{h(jcs)}^{(2)}(\gamma) = \frac{h_{20}}{\varepsilon^2} \left(\frac{\mu^2}{4E_\eta^2} \right)^\varepsilon \left(F_j^{(1)}(\gamma) + F_c^{(1)}(\gamma) + F_s^{(1)}(\gamma) \right). \quad (4.64)$$

In contrast, all remaining combinations of regions can possibly develop up to two poles in the rapidity regulator α . Then the pole structure for the mixed terms with genuine two-loop coefficients reads

$$\begin{aligned} F_{jc}^{(2)}(\gamma) &\simeq \left\{ \frac{(jc)_{40}}{\varepsilon^4} + \frac{(jc)_{31}}{\alpha \varepsilon^3} + \frac{(jc)_{22}}{\alpha^2 \varepsilon^2} \right\} \left(\frac{\mu^2}{2m_\eta E_\eta} \right)^\varepsilon \left(\frac{\mu^2}{m_\eta^2} \right)^\varepsilon \left(\frac{v}{2E_\eta} \right)^\alpha, \\ F_{js}^{(2)}(\gamma) &\simeq \left\{ \frac{(js)_{40}}{\varepsilon^4} + \frac{(js)_{31}}{\alpha \varepsilon^3} + \frac{(js)_{22}}{\alpha^2 \varepsilon^2} \right\} \left(\frac{\mu^2}{2m_\eta E_\eta} \right)^\varepsilon \left(\frac{\mu^2}{m_\eta^2} \right)^\varepsilon \left(\frac{v}{m_\eta} \right)^\alpha, \\ F_{cs}^{(2)}(\gamma) &\simeq \left\{ \frac{(cs)_{40}}{\varepsilon^4} + \frac{(cs)_{31}}{\alpha \varepsilon^3} + \frac{(cs)_{22}}{\alpha^2 \varepsilon^2} \right\} \left(\frac{\mu^2}{m_\eta^2} \right)^{2\varepsilon} \left(\frac{v}{2E_\eta} \right)^\alpha \left(\frac{v}{m_\eta} \right)^\alpha. \end{aligned} \quad (4.65)$$

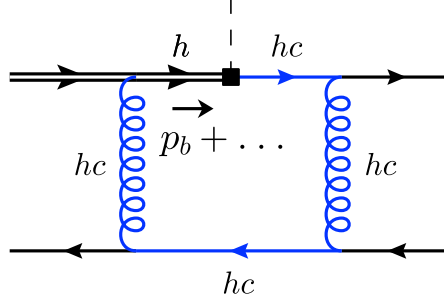


Figure 4.2: An example hard-collinear loop. Propagators with hard-collinear momentum scaling are depicted in blue. The dots refer to any momentum that can be obtained by shifts of the loop momentum, as long as p_b flows exclusively through the heavy-quark line.

In (4.63) and (4.65) we defined a total of 17 unknown coefficients. After evaluating all conditions given by the constraint that all poles vanish, we are able to express the final result for the double logarithm at the two-loop level

$$F^{(2)}(\gamma) \simeq \left(6h_{40}F^{(0)}(\gamma) + \frac{j_{40}}{6} - \frac{cs_{22}}{24} - \frac{7}{6}(h_{20})^2F^{(0)}(\gamma) + \frac{5}{6}h_{20}j_{20} \right) L^4, \quad (4.66)$$

in terms of only three genuine two-loop coefficients: h_{40} , j_{40} and cs_{22} . The hard coefficient follows from exponentiation (cf. (4.38))

$$h_{40} = \frac{(-C_F)^2}{2}, \quad (4.67)$$

and the mixing term between the soft and collinear regions has been calculated in [44] (see (D.18) therein) and reads

$$(cs)_{22} = \xi_0 C_F \left(-16C_F \frac{1+\bar{u}_0}{\bar{u}_0^3} + 2C_A \frac{2+\bar{u}_0}{\bar{u}_0^3} \right). \quad (4.68)$$

This leaves us with only one genuine two-loop coefficient to be calculated: j_{40} , i. e. the leading ε pole of the hard-collinear region. We choose this coefficient because its determination is free from ambiguities regarding the implementation of rapidity regulators. In the hard-collinear region all divergences are regulated by dimensional regularization. In the next section we describe how we determine this missing coefficient by means of an automated loop calculation to finalize the independent check of the integral equations (3.126) at the two-loop level.

4.3 CALCULATION OF j_{40}

In this section we describe the calculation of j_{40} in some more detail. We discuss the automated setup that has been used to perform the calculation, and as a last step we use the result to finalize the check of the integral equations (3.126).

The aim of this section is to compute the two-loop equivalent of the leading poles of the one-loop hard-collinear correction to the form factor $F(\gamma)$ in (4.42). In the previous section we obtained this by considering the one-loop corrections to the hard-collinear matching of the SCET-II operators in (4.8) and (4.13) and setting all other ingredients of the factorization formula to their tree-level values. However, this strategy introduces an unnecessary layer of complication, because we are

j_{20}^A	$\bar{u}_0^0$	$\bar{u}_0^{-1}$	$\bar{u}_0^{-2}$	$\bar{u}_0^{-3}$	$\bar{u}_0^{-4}$
j_{20}^{CA}	$1.0 \cdot 10^{-9}$	$-1.8 \cdot 10^{-9}$	$1.1 \cdot 10^{-9}$	-1.00000000026	$2.3 \cdot 10^{-11}$
j_{20}^{CF}	$7.3 \cdot 10^{-9}$	$-1.3 \cdot 10^{-9}$	8.00000000075	9.9999999981	$1.6 \cdot 10^{-10}$

Table 4.1: Numerical results for the one-loop leading ε -pole of the hard-collinear region j_{20} . We give the first two digits that differ from an (half-)integer.

only interested in the total hard-collinear contribution and not in the separate matching coefficients themselves. For this reason we pursue a different strategy in this section. Namely, we compute the (purely) hard-collinear region of the corresponding loop diagrams. The results of separate regions are gauge invariant and we choose Feynman gauge for the calculation presented in this section. This implies that we expect double-pole contributions from a larger set of diagrams than those in our hybrid gauge. To be clear, we include all diagrams in this section, including even diagrams that clearly do not contribute at the double-logarithmic level, e. g. self-energies.

In the following we describe the strategy of the computation and the employed automatic setup. We discuss the calculation of the one-loop leading hard-collinear pole j_{20} along with j_{40} . The overall framework of the developed code is based on `Alibary` [143] with several modifications where appropriate. As a first step we generate all required diagrams with the help of `QGRAF` [144], find 31 diagrams at the one-loop level and 722 diagrams at the two-loop level, and insert the corresponding Feynman rules. In order to obtain an identification of the regions consistent with the rest of the analysis, we have to fix the momentum routing prior to expanding in the hard-collinear region such that the hard external momentum of the b quark flows exclusively through the b quark propagators (see Fig. 4.2). Once this is achieved the remaining momentum flow can be left arbitrary, since the hard-collinear region is invariant under momentum shifts that have hard-collinear, collinear or soft scaling. The output is then passed to `FORM` [145, 146] which handles the next steps. We use `FORM`'s own function `trace4` and `color.h` [147] to simplify the Dirac structure in four dimensions and the color structure. It is sufficient to solve the Dirac structure in four dimensions since we are only interested in the leading ε -pole. We decompose all momenta into light-cone coordinates (cf. (1.9)) and expand to leading power in the power counting of the hard-collinear region (see Table 1.1). We pass the output back to `Mathematica`, identify all occurring loop integrals and determine the loop topologies (see Sec. 1.3). At one-loop there are 34 topologies and at two-loop 853. The large number of topologies does not turn out to be a bottleneck and we therefore do not attempt to minimize it, which could be done by finding symmetries among topologies that can be resolved by momentum shifts.³ The integrals are then passed to `FIRE6` [86] which performs the IBP reduction. At one-loop there are five MIs

$$\begin{aligned}
B_1 &= \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k_- (k^2 + p_{2-} k_+) (k^2 - \ell_+ (k_- + p_{\eta-}) + k_+ p_{\eta-})}, \\
B_2 &= \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{(k^2 + k_+ p_{1-}) (k^2 - \ell_+ (k_- + p_{\eta-}) + k_+ p_{\eta-})}, \\
B_3 &= \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k_- (k^2 + k_+ p_{1-}) (k^2 - \ell_+ (k_- + p_{\eta-}) + k_+ p_{\eta-})},
\end{aligned} \tag{4.69}$$

³ In an alternative setup implemented by one of my collaborators, this step was included leading to a reduction in the number of topologies by a factor of two. We note, however, that this does not affect the number of MIs.

j_{40}^A	$\bar{u}_0^0$	$\bar{u}_0^{-1}$	$\bar{u}_0^{-2}$	$\bar{u}_0^{-3}$	$\bar{u}_0^{-4}$
$j_{40}^{C_A^2}$	$-2.2 \cdot 10^{-7}$	$4.7 \cdot 10^{-7}$	$-3.6 \cdot 10^{-7}$	$1.2 \cdot 10^{-7}$	$1.5 \cdot 10^{-8}$
$j_{40}^{C_F C_A}$	$6.6 \cdot 10^{-7}$	$-1.4 \cdot 10^{-6}$	-0.4999989	-2.50000036	$4.3 \cdot 10^{-8}$
$j_{40}^{C_F^2}$	$-4.3 \cdot 10^{-7}$	$9.2 \cdot 10^{-7}$	14.99999929	17.00000023	$-2.8 \cdot 10^{-8}$

Table 4.2: Numerical results for the two-loop leading ε -pole of the hard-collinear region j_{40} . We give the first two digits that differ from an (half-)integer.

$$B_4 = \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{(k^2 - k_+ p_{1-}) (k^2 - \ell_+ (k_- + p_{2-}) + p_{2-} k_+)},$$

$$B_5 = \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{(k^2 - k_+ p_{1-}) (k^2 - \ell_+ k_-)},$$

and at two-loops there are 88 MIs given in Appendix A. In (4.69) we identify $\ell_+ = \bar{u}_0 m_\eta$, $p_{\eta-} = 2\gamma m_\eta$, $p_{2-} = 2\gamma \bar{u}_0 m_\eta$, $p_{1-} = 2\gamma(1 - \bar{u}_0)m_\eta$ following our definitions in chapter 2 to arrive at a compact representation. Inspecting the MI it is straightforward to see that they only have a non-trivial dependence on one parameter, which is $\bar{u}_0$. Since we are calculating the leading poles only, we expect that the final result for $j_{20/40}$ is given in terms of integers or simple rational numbers. It should therefore be sufficient to perform a numerical analysis of the MIs allowing to straightforwardly read off the analytic result. To resolve the functional dependence on $\bar{u}_0$ we evaluate the MIs numerically for a set of points in $\bar{u}_0$

$$\bar{u}_0 \in \left\{ \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10} \right\}. \quad (4.70)$$

The numerical analysis of the MIs is performed using pySecDec [148] in combination with the built-in quasi-Monte Carlo integrator [149]. The next step is then to perform a separate fit to the $\bar{u}_0$ -dependence for every possible color factor A , which we parameterize as

$$j_{20/40}(\bar{u}_0) = \xi_0 \sum_A \sum_{n=0}^4 A j_{20/40,n}^A \bar{u}_0^{-n}. \quad (4.71)$$

At the one-loop level the result of the fit is given in Table 4.1. Given the numerical precision of the fit we refrain from performing a detailed numerical analysis of the related uncertainties and identify the analytic result

$$j_{20} = \frac{\xi_0}{\bar{u}_0^3} (C_F(10 + 8\bar{u}_0) - C_A), \quad (4.72)$$

which matches with the previously obtained result in (4.62). At the two-loop level the output of the fit is given in Table 4.2. In this case the numerical precision is slightly worse compared to the one-loop fit due to increasing cancellations at the two-loop level, but still sufficiently high to justify to proceed in a similar fashion as at the one-loop level. Reading off the analytic result yields

$$j_{40} = \frac{\xi_0 C_F}{\bar{u}_0^3} \left(C_F (17 + 15\bar{u}_0) + C_A \left(-\frac{5}{2} - \frac{\bar{u}_0}{2} \right) \right), \quad (4.73)$$

which consists of a set of non-trivial coefficients. Inserting (4.73) into the master formula for the overall double-logarithmic contribution at the two-loop level in (4.66) yields

$$F^{(2)}(\gamma) \simeq \xi_0 L^4 \left(-C_F^2 \frac{7 + 10\bar{u}_0}{6\bar{u}_0^3} + C_A C_F \frac{3 - 2\bar{u}_0}{12\bar{u}_0^3} \right), \quad (4.74)$$

which confirms the two-loop result predicted from the integral equations in (3.126) that we derived in a diagrammatic analysis (cf. (3.52)) in chapter 3. This is a remarkable result. In particular, it confirms the iterative nature of the integral equations and the exponential Sudakov factor as part of the integrand as we discussed in Sec. 4.2.2.

We now study the iterative integral equations (3.126) derived in chapter 3 and confirmed up to the two-loop order in the previous chapter. We give a recursive definition of the coefficients of the large logarithms L^{2n} defined in (3.21) and determine the asymptotics of the two auxiliary functions f_1 and f_2 in the limit of infinite b -quark mass. To facilitate the discussion, we formulate relevant quantities in terms of logarithmic degrees of freedom. The final formula for the form factor (3.128) requires the on-shell auxiliary functions f_1 and f_2 for which we write

$$G_1(z) \equiv f_1(m_2, m_2), \quad G_2(z) \equiv f_2(m_2, m_2), \quad (5.1)$$

where

$$z = \frac{\alpha_s C_F}{2\pi} L^2. \quad (5.2)$$

For the integral equations (3.126) it is useful to define the logarithmic variables

$$\begin{aligned} \rho &= \sqrt{\frac{\alpha_s C_F}{2\pi}} \ln \left(\frac{q_+ p_-}{m_2^2} \right), & \eta &= \sqrt{\frac{\alpha_s C_F}{2\pi}} \ln \left(\frac{p_-}{q_-} \right), \\ \rho' &= \sqrt{\frac{\alpha_s C_F}{2\pi}} \ln \left(\frac{k_+ p_-}{m_2^2} \right), & \eta' &= \sqrt{\frac{\alpha_s C_F}{2\pi}} \ln \left(\frac{p_-}{k_-} \right). \end{aligned} \quad (5.3)$$

We express f_1 and f_2 in terms of ρ and η

$$f_1(q_+, q_-) = g_1(\rho, \eta), \quad f_2(q_+, q_-) = g_2(\rho, \eta). \quad (5.4)$$

Similarly, the integral equations (3.126) become

$$\begin{aligned} g_1(\rho, \eta) &= 1 + \int_0^\eta d\eta' \int_{\eta'}^\rho d\rho' e^{-\eta'(\rho-\rho')} g_1(\rho', \eta'), \\ g_2(\rho, \eta) &= 1 + \int_0^\eta d\eta' \int_{\eta'}^\rho d\rho' e^{-\eta'(\rho-\rho')} \left[\left(g_2(\rho', \eta') + \frac{1}{2} \right) - \frac{1}{4N_c C_F} (g_1(\rho', \eta') - 1) \right], \end{aligned} \quad (5.5)$$

where we wrote C_A as $C_A = 2C_F + 1/N_c$. Decomposing the color structure in this way, it is immediately clear that mixing between f_1 (g_1) and f_2 (g_2) is suppressed by $\sim 1/N_c^2$ in the large N_c limit.

Before we analyze the integral equations (5.5) in more detail, we compare them to integral equations that were found for other processes where the respective factorization theorem also suffers from endpoint-divergent convolution integrals.

- (i) The first process we consider is the bottom-induced Higgs decay to photons [35, 150, 151]. Here the large hierarchy between the mass of the Higgs and bottom quarks produces logarithms that need to be resummed. The Higgs transfers a large energy to the bottom quarks leading to an exponential Sudakov factor. But in this case there appears only a single soft-quark propagator, because additional soft-quark propagators are power suppressed. Hence,

the double logarithms are resummed by a non-iterative two-fold integral with an exponential Sudakov in the integrand. Adapting our notation, the double-logarithmic amplitude reads

$$\mathcal{A}^{(b)}(h \rightarrow \gamma\gamma) \simeq \frac{2\pi\mathcal{A}_0^{(b)}}{\alpha_s C_F} \int_0^\eta d\eta' \int_{\eta'}^\rho d\rho' e^{-\eta'(\rho-\rho')} , \quad \rho = \eta = \sqrt{\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{m_H^2}{m_b^2}} . \quad (5.6)$$

The integral can be solved and gives a generalized hypergeometric function

$$\int_0^\rho d\eta' \int_{\eta'}^\rho d\rho' e^{-\eta'(\rho-\rho')} = \frac{\rho^2}{2} {}_2F_2\left(1, 1; \frac{3}{2}, 2; -\frac{\rho^2}{4}\right) . \quad (5.7)$$

- (ii) The second example is muon-electron backward scattering [78], which is governed by a single form factor f_1 in the double-logarithmic approximation. In this case there is no *external* momentum transfer and, thus, there is no exponential Sudakov factor. In contrast to bottom-induced Higgs decays to photons, here, additional soft lepton propagators are not power suppressed and therefore the integral equations that describe the double-logarithmic series are iterative

$$f_1(\rho, \eta) = 1 + \int_0^\eta d\eta' \int_{\eta'}^\rho d\rho' f_1(\rho', \eta') . \quad (5.8)$$

The solution can be expressed in terms of a modified Bessel function

$$f_1(\rho, \rho) = \frac{I_1(2\rho)}{\rho} , \quad \text{with } \rho = \sqrt{\frac{\alpha}{2\pi} \ln^2 \frac{s}{m^2}} , \quad (5.9)$$

where m is the lepton mass that is taken to be of the same order for muons and electrons, and $s = 4E^2$ is the center-of-mass energy.

The process we consider has both features: iterative soft-quark propagators and external momentum transfer leading to an exponential Sudakov factor. The combination of both sources of double logarithms is a new feature in comparison to the examples (i) and (ii) and makes a closed-form solution more challenging.

5.1 ONE-FOLD INTEGRAL EQUATIONS FOR ON-SHELL FORM FACTORS

Rather than solving the integral equations directly, we recast them as differential equations, aiming to reduce them to one-fold integral equations that are more tractable for an order-by-order analysis. The first step is then to derive the differential equations for the auxiliary functions g_1 and g_2 . Performing the derivative with respect to η of the defining equation for g_1 in (5.5) yields

$$\partial_\eta g_1(\rho, \eta) = \int_\eta^\rho d\rho' e^{-\eta(\rho-\rho')} g_1(\rho', \eta) . \quad (5.10)$$

Since in (5.10) also the integrand depends on ρ the derivative of (5.10) with respect to ρ has multiple terms

$$\partial_\rho \partial_\eta g_1(\rho, \eta) = g_1(\rho, \eta) + \int_\eta^\rho d\rho' e^{-\eta(\rho-\rho')} (-\eta) g_1(\rho', \eta) . \quad (5.11)$$

From (5.11) and (5.10) follows the partial differential equation

$$(\partial_\rho \partial_\eta + \eta \partial_\eta - 1) g_1(\rho, \eta) = 0 . \quad (5.12)$$

The appropriate boundary conditions for (5.12) can be deduced from the integral equations (5.5)

$$g_1(\rho, \eta = 0) = 1, \quad \partial_\eta g_1(\rho, \eta) \Big|_{\rho=\eta} = 0. \quad (5.13)$$

Following an equivalent set of steps for g_2 leads to the differential equation

$$(\partial_\rho \partial_\eta + \eta \partial_\eta - 1) \left(g_2(\rho, \eta) + \frac{1}{2} \right) = -\frac{1}{4N_c C_F} (g_1(\rho, \eta) - 1), \quad (5.14)$$

and corresponding boundary conditions

$$g_2(\rho, \eta = 0) = 1, \quad \partial_\eta g_2(\rho, \eta) \Big|_{\rho=\eta} = 0. \quad (5.15)$$

Although the differential equations for g_1 and g_2 may appear straightforward at first glance, they cannot be solved in closed form. The problem is that with standard mathematical methods, e. g. a transformation to Laplace space, the solutions cannot be combined with the boundary conditions (5.13) and (5.15) in a consistent way. As an alternative, we transform the differential equations (5.12) and (5.14) to one-fold integral equations for *on-shell* form factors that allow for a simpler order-by-order solution compared to the original integral equations (5.5). To this end, we notice that the differential equations (5.12) and (5.14) take a simpler form if we perform another derivative with respect to η

$$\begin{aligned} (\partial_\rho + \eta) \partial_\eta^2 g_1(\rho, \eta) &= 0, \\ (\partial_\rho + \eta) \partial_\eta^2 g_2(\rho, \eta) &= -\frac{1}{4N_c C_F} \partial_\eta g_1(\rho, \eta). \end{aligned} \quad (5.16)$$

Focusing for now on g_1 , we define $h_1(\rho, \eta) = \partial_\eta^2 g_1(\rho, \eta)$. The resulting differential equation for h_1

$$(\partial_\rho + \eta) h_1(\rho, \eta) = 0, \quad (5.17)$$

is trivially solved by

$$h_1(\rho, \eta) = \tilde{C}_1(\eta) e^{-\rho\eta}. \quad (5.18)$$

In particular, the differential equation for h_1 (5.17) fixes only the dependence on ρ and the boundary condition may depend on η . Our aim in the following is to relate $\tilde{C}_1(\eta)$ to g_1 . Integrating the defining equation for h_1 yields

$$\int_\rho^{\eta'} d\eta'' \partial_{\eta''}^2 g_1(\rho, \eta'') = \partial_{\eta'} g_1(\rho, \eta') - \partial_{\eta'} g_1(\rho, \eta') \Big|_{\eta'=\rho} = \partial_{\eta'} g_1(\rho, \eta'), \quad (5.19)$$

and

$$\int_0^\eta d\eta' \partial_{\eta'} g_1(\rho, \eta') = g_1(\rho, \eta) - g_1(\rho, 0) = g_1(\rho, \eta) - 1, \quad (5.20)$$

where we chose the integration domains such that the boundary conditions (5.13) and (5.15) are applicable. Combining (5.18), (5.19) and (5.20) yields

$$g_1(\rho, \eta) = 1 + \int_0^\eta d\eta' \int_{\eta'}^\rho d\eta'' e^{-\eta''(\rho-\eta'')} C_1(\eta''), \quad (5.21)$$

where we defined

$$\tilde{C}_1(\eta) = -e^{\eta^2} C_1(\eta). \quad (5.22)$$

For later convenience, we convert the integrals in (5.21) to one-fold integrals. Splitting the second integral in (5.21) gives

$$\int_0^\eta d\eta' \int_{\eta'}^\rho d\eta'' e^{-\eta''(\rho-\eta'')} C_1(\eta'') = \left[\int_0^\eta d\eta' \int_{\eta'}^\eta d\eta'' + \int_0^\eta d\eta' \int_\eta^\rho d\eta'' \right] e^{-\eta''(\rho-\eta'')} C_1(\eta''). \quad (5.23)$$

The η' -integral in the first term can be manipulated by means of integration-by-parts identities and in the second term the η' -integral is trivial since the integrand does not depend on η' . This allows us to write

$$g_1(\rho, \eta) = 1 + \left(\int_0^\eta d\eta' \eta' + \int_\eta^\rho d\eta' \eta \right) e^{-\eta'(\rho-\eta')} C_1(\eta'). \quad (5.24)$$

Plugging (5.21) into a rearranged version of the differential equation (5.12) gives

$$\begin{aligned} g_1(\rho, \eta) &= (\partial_\rho + \eta) \partial_\eta g_1(\rho, \eta) \\ &= C_1(\rho) + \int_\eta^\rho d\eta' (\eta - \eta') e^{-\eta'(\rho-\eta')} C_1(\eta'). \end{aligned} \quad (5.25)$$

In particular, (5.25) implies that $C_1(\rho)$ corresponds to the on-shell version of g_1

$$g_1(\rho, \rho) = C_1(\rho) = G_1(\rho^2). \quad (5.26)$$

Equating the expressions for $g_1(\rho, \eta)$ in (5.24) and (5.25), we obtain a one-fold integral equation for the auxiliary on-shell function $C_1(\rho)$

$$C_1(\rho) = 1 + \int_0^\rho d\rho' \rho' e^{-\rho'(\rho-\rho')} C_1(\rho'). \quad (5.27)$$

We note that in contrast to the original two-fold integral equation for $g_1(\rho, \eta)$ in (5.5), we cannot convert (5.27) into a differential equation. The reason is that a derivative with respect to ρ gives additional powers of ρ' in the integrand and the resulting integral cannot be expressed in terms of $C_1(\rho)$. For later convenience we bring the integral equation (5.27) into an alternative but equivalent form. Renaming $\rho \rightarrow \rho''$ and integrating ρ'' from 0 to ρ gives for the integral in (5.27)

$$\begin{aligned} \int_0^\rho d\rho'' \int_0^{\rho''} d\rho' \rho' e^{-\rho'(\rho''-\rho')} C_1(\rho') &= \int_0^\rho d\rho' \int_{\rho'}^\rho d\rho'' \rho' e^{-\rho'(\rho''-\rho')} C_1(\rho') \\ &= \int_0^\rho d\rho' \left(1 - e^{-\rho'(\rho-\rho')} \right) C_1(\rho'), \end{aligned} \quad (5.28)$$

where we changed the order of integration in a way that respects the integration domain in the first step such that the ρ'' -integral becomes trivial. Performing the same integral on the remaining terms in (5.27) leads to the integrated one-fold integral equation

$$\rho = \int_0^\rho d\rho' e^{-\rho'(\rho-\rho')} C_1(\rho'). \quad (5.29)$$

We employ a power series ansatz

$$C_1(\rho) = \sum_{n=0}^{\infty} a_n (-\rho^2)^n, \quad (5.30)$$

to solve the integral equation (5.27) order-by-order. Indeed, decomposing the integral in (5.27) into series yields

$$\begin{aligned} \int_0^\rho d\rho' \rho' e^{-\rho'(\rho-\rho')} C_1(\rho') &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_k}{m!} \int_0^\rho d\rho' \rho' (-\rho'(\rho-\rho'))^m (-\rho'^2)^k \\ &= - \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_k}{m!} (-\rho^2)^{1+k+m} \frac{m!(1+m+2k)!}{(2+2m+2k)!}. \end{aligned} \quad (5.31)$$

In our ansatz (5.30) for $C_1(\rho)$ we implicitly assumed that the solution only contains even powers of ρ . If, for the moment, we allow the sum over k in (5.31) to include half-integer k corresponding to odd powers of ρ , we see immediately that half-integer k lead to odd powers of ρ also upon integration. Thus, odd powers of ρ decouple and form a self-contained system, i. e. they can only be produced by lower order odd powers. But inspecting the defining integral equation (5.27) we see that there is no linear term and we conclude that also all other odd powers of ρ vanish. This outcome is expected, given that ρ is a logarithmic variable and $C_1(\rho)$ is a double-logarithmic series. Isolating the coefficient of $(-\rho^2)^n$ in (5.31) we can fix one of the sums by imposing $1+k+m=n$. We choose to constrain the sum over m and replace $m=n-k-1$ in (5.31). Hence, the coefficient a_n of $(-\rho^2)^n$ reads

$$a_n = - \sum_{k=0}^{n-1} \frac{(n+k)!}{(2n)!} a_k, \quad (n \geq 1). \quad (5.32)$$

For concreteness, the first orders of the expansion are given by

$$C_1(\sqrt{z}) = G_1(z) = \sum_{n=0}^{\infty} a_n (-z)^n = 1 + \frac{z}{2} + \frac{z^2}{4!} - \frac{z^3}{6!} - \frac{z^4}{8!} + \frac{17z^5}{10!} + O(z^6), \quad (5.33)$$

where we used $\rho^2 \rightarrow z$ to obtain the on-shell result.

The differential equations for g_2 are more complicated due to their inhomogeneity. It is convenient to decompose g_2 in terms of its two color structures

$$g_2(\rho, \eta) \equiv g_{2a}(\rho, \eta) + \frac{1}{4N_c C_F} g_{2b}(\rho, \eta). \quad (5.34)$$

In analogy to the steps that led to the original differential equations (5.12) and (5.14) we derive separate differential equations for both color structures

$$\begin{aligned} (\partial_\rho \partial_\eta + \eta \partial_\eta - 1) \left(g_{2a}(\rho, \eta) + \frac{1}{2} \right) &= 0, \\ (\partial_\rho \partial_\eta + \eta \partial_\eta - 1) g_{2b}(\rho, \eta) &= -(g_1(\rho, \eta) - 1), \end{aligned} \quad (5.35)$$

with boundary conditions

$$g_{2a}(\rho, 0) = 1, \quad g_{2b}(\rho, 0) = 0, \quad (5.36)$$

$$\partial_\eta g_{2a}(\rho, \eta) \Big|_{\rho=\eta} = 0, \quad \partial_\eta g_{2b}(\rho, \eta) \Big|_{\rho=\eta} = 0. \quad (5.37)$$

The differential equation for g_{2a} can be solved in a manner analogous to that of g_1 . The steps leading to the equivalent of (5.24) remain unchanged. For (5.25), on the other hand, the left hand side must be modified to $g_{2a} + 1/2$. To equate the equivalents of (5.24) and (5.25) we have to add $1/2$ to (5.24). Consequently, the integral equation for C_{2a} is

$$C_{2a}(\rho) = \frac{3}{2} + \int_0^\rho d\rho' \rho' e^{-\rho'(\rho-\rho')} C_{2a}(\rho'), \quad (5.38)$$

where C_{2a} is defined in analogy to C_1 . From the equivalent of (5.25) it follows that

$$g_{2a}(\rho, \rho) + \frac{1}{2} = C_{2a}(\rho). \quad (5.39)$$

Combining (5.39) and a comparison of the one-fold integral equations for C_1 and C_{2a} implies

$$g_{2a}(\rho, \rho) \equiv G_{2a}(\rho^2) = \frac{3}{2} G_1(\rho^2) - \frac{1}{2}. \quad (5.40)$$

Hence, $G_{2a}(\rho^2)$ and $G_1(\rho^2)$ are closely related, as expected due to the similarity of their integral equations. The differential equation for g_{2b} is more involved due to its inhomogeneity

$$(\partial_\rho + \eta) \partial_\eta^2 g_{2b}(\rho, \eta) = -\partial_\eta g_1(\rho, \eta). \quad (5.41)$$

In analogy to the derivation of (5.21) and implementing the boundary conditions (5.36), we obtain

$$g_{2b}(\rho, \eta) = \int_0^\eta d\eta' \int_{\eta'}^\rho d\eta'' e^{-\eta''(\rho-\eta'')} C_{2b}(\rho, \eta''), \quad (5.42)$$

where we defined $C_{2b}(\rho, \eta)$ as

$$\partial_\eta^2 g_{2b}(\rho, \eta) = -C_{2b}(\rho, \eta) e^{-\eta(\rho-\eta)}. \quad (5.43)$$

Here $C_{2b}(\rho, \eta)$ depends on both ρ and η because the residual ρ -dependence has to account for the inhomogeneous term in contrast to $C_1(\rho)$, since there is no inhomogeneity in this case. For concreteness, the ρ -dependence has to obey

$$-\partial_\eta g_1(\rho, \eta) = -e^{-\eta(\rho-\eta)} \partial_\rho C_{2b}(\rho, \eta) \quad (5.44)$$

such that (5.41) is fulfilled. Integrating (5.44) over the appropriate domain gives the ρ -dependence of $C_{2b}(\rho, \eta)$

$$\begin{aligned} C_{2b}(\rho, \eta) &= C_{2b}(\eta, \eta) + \int_\eta^\rho d\rho' e^{-\eta(\eta-\rho')} \partial_\eta g_1(\rho', \eta) \\ &= C_{2b}(\eta, \eta) + \int_\eta^\rho d\eta' \int_\eta^\rho d\rho' e^{-\eta(\eta-\rho')} e^{-\eta'(\rho'-\eta')} C_1(\eta'), \end{aligned} \quad (5.45)$$

where $C_{2b}(\eta, \eta)$ can be viewed as a boundary condition that has yet to be determined. Rearranging the differential equations (5.41) yields

$$\begin{aligned} g_{2b}(\rho, \eta) &= (\partial_\rho + \eta) \partial_\eta g_{2b}(\rho, \eta) + (g_1(\rho, \eta) - 1) \\ &= C_{2b}(\rho, \rho) + \int_\eta^\rho d\eta' (\eta - \eta') e^{-\eta'(\rho-\eta')} C_{2b}(\rho, \eta') + (g_1(\rho, \rho) - 1), \end{aligned} \quad (5.46)$$

where we used (5.42) and (5.44) in intermediate steps. Setting $\rho = \eta$ in (5.46) we can relate the on-shell quantities of the $1/N_c^2$ suppressed color structure to $C_1(\rho)$

$$g_{2b}(\rho, \rho) \equiv G_{2b}(\rho^2) = C_{2b}(\rho, \rho) + (C_1(\rho) - 1) . \quad (5.47)$$

Deriving a one-fold integral equation for $g_{2b}(\rho, \rho)$ proceeds similarly to the derivation of (5.27). After some algebra, which can be found in Appendix B, we obtain

$$1 + g_{2b}(\rho, \rho) = 1 + \int_0^\rho d\rho' \rho' e^{-\rho'(\rho-\rho')} \left(1 + g_{2b}(\rho', \rho') \right) + \int_0^\rho d\rho' \rho' \mathcal{K}(\rho'(\rho-\rho')) C_1(\rho') , \quad (5.48)$$

where the integration kernel $\mathcal{K}(x)$ is defined as

$$\mathcal{K}(x) = \int_0^\infty dx' \left[\frac{\theta(x-x')}{x(x-x')} \right]_+ x' e^{-x'} , \quad (5.49)$$

and the plus distribution (cf. (1.111)) subtracts the limit $x' \rightarrow x$. Repeating the steps that lead to the integrated one-fold integral equation for C_1 in (5.29) on (5.48) gives

$$\rho = \int_0^\rho d\rho' \rho' e^{-\rho'(\rho-\rho')} \left(1 + g_{2b}(\rho', \rho') \right) + \int_0^\rho d\rho' \rho' \mathcal{K}_I(\rho'(\rho-\rho')) C_1(\rho') , \quad (5.50)$$

where $\mathcal{K}_I$ is defined as

$$\mathcal{K}_I(x) = - \int_0^x dx' \mathcal{K}(x') . \quad (5.51)$$

This completes the derivation of the one-fold integral equation for $g_{2b}(\rho, \rho)$ and allows us to compute the expansion of $g_{2b}(\rho, \rho)$ order-by-order in ρ in analogy to $C_1(\rho)$. It is convenient to start with the expansion of $\mathcal{K}(x)$

$$\begin{aligned} \mathcal{K}(x) &= \int_0^x dx' \frac{x' e^{-x'} - x e^{-x}}{x(x-x')} \\ &= \frac{1}{x} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^x dx' \left[\frac{(x')^{m+1} - x^{m+1}}{x-x'} \right] . \end{aligned} \quad (5.52)$$

The integral evaluates to

$$\int_0^x dx' \left[\frac{(x')^{m+1} - x^{m+1}}{x-x'} \right] = -x^{m+1} H_{m+1} , \quad (5.53)$$

where H_n is the n -th harmonic number. Hence, the expansion of the last term in (5.48) is given by

$$\begin{aligned} \int_0^\rho d\rho' \rho' \mathcal{K}(\rho'(\rho-\rho')) C_1(\rho') &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} a_k \frac{(-1)^{m+1}}{m!} H_{m+1} \int_0^\rho d\rho' \rho' (-\rho'^2)^k (\rho'(\rho-\rho'))^m \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} a_k H_{m+1} (-\rho^2)^{1+k+m} \frac{(1+2k+m)!}{(2+2k+2m)!} . \end{aligned} \quad (5.54)$$

We obtain the n -th power of $(-\rho^2)$ by imposing $n = 1 + k + m$, which fixes the sum over m ,

$$\left[\int_0^\rho d\rho' \rho' \mathcal{K}(\rho'(\rho - \rho')) C_1(\rho') \right]_{(-\rho^2)^n} = (-\rho^2)^n \sum_{k=0}^{n-1} a_k H_{n-k} \frac{(n+k)!}{(2n)!}. \quad (5.55)$$

Defining

$$1 + g_{2b}(\rho, \rho) = \sum_{n=0}^{\infty} b_n (-\rho^2)^n, \quad (5.56)$$

the expansion of the first term in (5.48) proceeds exactly analogous to the expansion of the one-fold integral equations for $C_1(\rho)$ (5.27) in (5.32). Consequently, the recursive equation for b_n becomes

$$b_n = - \sum_{k=0}^{n-1} \frac{(n+k)!}{(2n)!} (b_k - H_{n-k} a_k), \quad (n \geq 1). \quad (5.57)$$

Setting ρ on-shell, $\rho^2 = z$, the first few terms of the expansion read

$$1 + G_{2b}(z) = \sum_{n=0}^{\infty} b_n (-z)^n = 1 - 2 \left(\frac{z^2}{4!} + \frac{z^3}{6!} - \frac{7z^4}{8!} + \frac{23z^5}{10!} \right) + \mathcal{O}(z^6). \quad (5.58)$$

The result for $G_2(z)$ with full color information is given by

$$G_2(z) = G_{2a}(z) + \frac{1}{4N_c C_F} G_{2b}(z). \quad (5.59)$$

We now have all pieces at hand to construct a systematic expansion of the form factor $F(\gamma)$ in (3.126). We find agreement with our previously computed results at LO, NLO and NNLO in (3.2), (3.21) and (3.52), respectively. Furthermore, we are now in a position to make a prediction for the three-loop term

$$F^{(3)}(\gamma) \simeq \xi_0 L^6 \left(C_F^3 \frac{29 + 44\bar{u}_0}{90\bar{u}_0^3} - C_F^2 C_A \frac{1 - 28\bar{u}_0}{180\bar{u}_0^3} \right), \quad (5.60)$$

and higher-order terms.

5.2 ASYMPTOTIC BEHAVIOUR IN THE LARGE-RECOIL LIMIT

So far we were assuming that z defined in (5.2) is a quantity of order $\mathcal{O}(1)$. This implies that it is not subject to an expansion and must be resummed to all orders for a consistent treatment. It is also conceptually interesting to understand how this series behaves in the limit where z is large. Keeping in mind the definition of z in (5.2), L in (3.21) and the recoil γ in (2.4) we write z explicitly as

$$z = \frac{\alpha_s C_F}{2\pi} \ln^2 \left(\frac{m_B}{m_\eta} \right), \quad (5.61)$$

to see that the limit of large z corresponds to the limit of large b -quark mass, if we keep m_η fixed. While z is not a large quantity given the physical values of the masses m_B and m_η , the limit of large b quark mass is formally the limit that defines HQET. It is therefore interesting to explore the convergence of this limit. Given that we do not have a closed-form solution at hand, we perform

a method-of-regions analysis of the one-fold integral equations in (5.27) and (5.48) to derive the analytic form of the soft-overlap form factor $F(\gamma)$ in the large-recoil limit.¹

We start with the expansion of the integral equation for $C_1(\rho)$ in (5.29). Here, we choose to work with the integrated version of the integral equation and discuss below why this version is more convenient in this case. As a first step it is useful to analyze the integrand. It is important to realize that the exponential leads to a suppression of the integrand in most of the integration domain. For a substantial contribution to the integral the argument of the exponential

$$e^{-\rho'(\rho-\rho')}, \quad (5.62)$$

must be of order $O(1)$. This is the case if either $\rho' \sim \rho^{-1}$ or $\rho - \rho' \sim \rho^{-1}$. To facilitate the method-of-regions analysis we formulate this in terms of regions

$$\begin{aligned} \text{region (A)} \quad \rho' &\sim \rho^{-1}, & \text{such that} \quad \rho'(\rho - \rho') &\approx \rho' \rho \sim O(1), \\ \text{region (B)} \quad \rho - \rho' &\sim \rho^{-1}, & \text{such that} \quad \rho'(\rho - \rho') &\approx \rho(\rho - \rho') \sim O(1), \end{aligned}$$

and define our power-counting parameter to be $\rho^{-1} \ll 1$. In region (A) we can expand $C_1(\rho')$ around 0 such that the integral in the integral equation (5.27) becomes

$$\frac{1}{\rho} \int_0^\rho d\rho' e^{-\rho'(\rho-\rho')} C_1(\rho') \stackrel{(A)}{=} \frac{C_1(0)}{\rho} \int_0^\infty d\rho' e^{-\rho\rho'} + O(z^{-2}) = \frac{G_1(0)}{z} + O(z^{-2}), \quad (5.63)$$

and $G_1(0) = 1$ is known from the series expansion of $G_1(z)$ in (5.33). We will see below that we do not need to expand this integral to higher powers in ρ^{-1} . In region (B) we introduce the variable $\tilde{\rho} = \rho - \rho' \sim \rho^{-1}$ to enable a straightforward power expansion

$$\begin{aligned} \frac{1}{\rho} \int_0^\rho d\rho' e^{-\rho'(\rho-\rho')} C_1(\rho') &= \frac{1}{\rho} \int_0^\rho d\tilde{\rho} e^{-\tilde{\rho}(\rho-\tilde{\rho})} C_1(\rho - \tilde{\rho}) \\ &\stackrel{(B)}{=} \frac{G_1(z)}{z} + \left(\frac{2G_1(z)}{z^2} - \frac{2G_1'(z)}{z} \right) + O\left(\frac{G_1(z)}{z^3} \right), \end{aligned} \quad (5.64)$$

where the power counting of $G_1(z)$ and its derivatives still has to be determined, and the bracket indicates power suppressed terms in the expansion in ρ^{-1} . Plugging (5.63) and (5.64) into the integral equation (5.29) yields an ordinary differential equation for $G_1(z)$

$$0 = -1 + \frac{1}{z} + \frac{G_1(z)}{z} + \left(\frac{2G_1(z)}{z^2} - \frac{2G_1'(z)}{z} \right) + O\left(\frac{G_1(z)}{z^3}, \frac{1}{z^2} \right). \quad (5.65)$$

We will now solve (5.65) order-by-order in ρ^{-1} and start with fixing the power counting of $G_1(z)$. If $G_1(z)$ has homogeneous power counting z^α , which we denote by $G_1(z) \sim z^\alpha$, with $\alpha > 1$, the term $G_1(z)/z$ in (5.65) is parametrically larger than other terms and the differential equation becomes at leading power

$$0 = \frac{G_1(z)}{z}, \quad (5.66)$$

which implies that these terms vanish and we conclude that $\alpha \leq 1$. The highest inhomogeneous term in (5.65) is $\sim \rho^0$. Assuming $G_1(z) \sim z^\alpha$ with $\alpha = 1$ means that $G_1(z)/z$ is also $\sim \rho^0$ and at this power the differential equations becomes

$$0 = -1 + \frac{G_1(z)}{z}. \quad (5.67)$$

¹ We note that the running of the strong coupling α_s is a single-logarithmic effect that can be neglected at the double-logarithmic level and it is therefore sufficient to evaluate α_s at a fixed scale.

Hence, $G_1(z) = z$ is the leading power expansion of $G_1(z)$ for large $z = \rho^2$. For power corrections we make the ansatz $G_1(z) = z + bz^\beta$. For $0 < \beta < 1$ the appropriate differential equation is still (5.67) and we obtain

$$0 = -1 + \frac{G_1(z)}{z} = bz^{\beta-1}. \quad (5.68)$$

Consequently, b must vanish.

At this point we compare the method-of-regions analysis of the integrated one-fold integral equation (5.29) with our initial one-fold integral equation (5.27). Following an equivalent set of steps that resulted in (5.65) leads to

$$0 = 1 + \frac{1}{z} + \left(\frac{G_1(z)}{z} - 2G_1'(z) \right) + \left(\frac{6G_1(z)}{z^2} - \frac{6G_1'(z)}{z} + 4G_1''(z) \right) + O\left(\frac{G_1(z)}{z^3}, \frac{1}{z^2} \right), \quad (5.69)$$

if we use the initial one-fold integral equation (5.27) as a starting point instead of the integrated one-fold integral equation (5.29). The differential equation in (5.69) is, however, incapable of constraining $\sqrt{z}$ terms in contrast to (5.68), which can be understood as follows. Modifying the integrated integral equations (5.29) by adding an arbitrary constant k on the l.h.s.

$$\rho + k = \int_0^\rho d\rho' e^{-\rho'(\rho-\rho')} C_1(\rho'), \quad (5.70)$$

and repeating the steps that lead to (5.68) gives

$$0 = -1 - \frac{k}{\sqrt{z}} + \frac{G_1(z)}{z} = bz^{\beta-1}, \quad (5.71)$$

which is solved by $b = k$ and $\beta = 1/2$ allowing for a $\sqrt{z}$ -term in the solution. The important point is that the initial or unintegrated integral equation is the derivative of the integrated integral equation with respect to ρ . Since the derivative of the constant k vanishes, the initial integral equation (5.27) is not able to tell whether the constant is present and therefore incapable of discriminating between both scenarios.

In terms of power counting the differential equation in (5.65), including also the terms in the bracket, is sufficient to constrain terms constant in z . The ansatz $G_1(z) = z + c$ leads to

$$0 = -1 + \frac{1}{z} + \frac{G_1(z)}{z} + \left(\frac{2G_1(z)}{z^2} - \frac{2G_1'(z)}{z} \right) = \frac{1}{z} + \frac{c}{z} + O(z^{-2}). \quad (5.72)$$

which is solved by $c = -1$ corresponding to

$$G_1(z) = z - 1. \quad (5.73)$$

In Fig. 5.1 we show a comparison of the asymptotic expansion (5.73) to a partial sum

$$G_1(z; N) \equiv \sum_{n=0}^N a_n(-z)^n \quad (5.74)$$

encoding the series in (5.33) valid for small values of z , where N is the power of the highest term in $(-z)$. We see that the asymptotic expansion works well until the series reaches values of z where higher z -terms than $(-z)^N$ start to contribute significantly, as expected. We find that this range can be extended by increasing N and show $N = 5, 15, 25$ as an example in Fig. 5.1. The range of validity

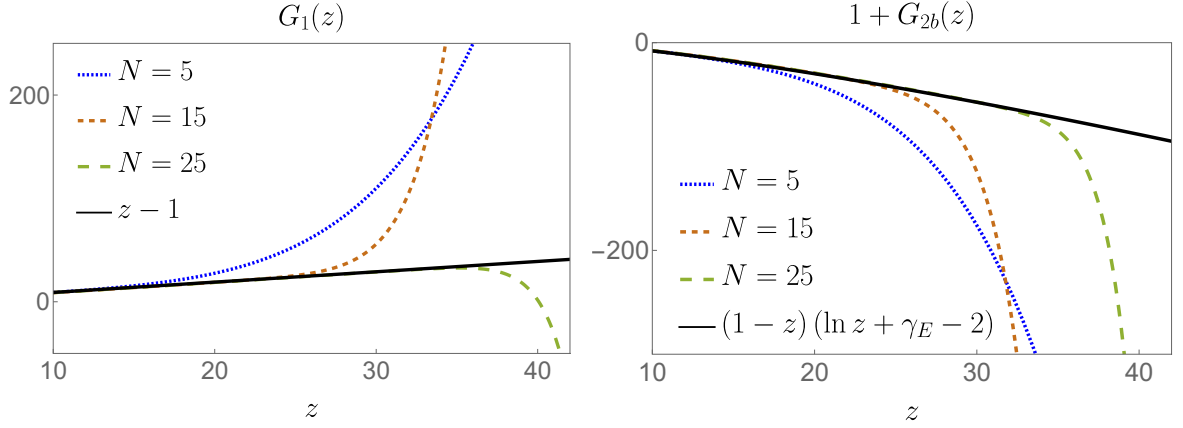


Figure 5.1: Partial sums of the two form factors $G_1(z; N) \equiv \sum_{n=0}^N a_n(-z)^n$ (left panel) and $1 + G_{2b}(z; N) \equiv \sum_{n=0}^N b_n(-z)^n$ (right panel) as a function of z defined in (5.2), and their convergence towards the asymptotic limit for large values of z , given in (5.73) and (5.92) (black solid curves). The recursion relations that determine the coefficients a_n and b_n to all orders can be found in (5.32) and (5.57), respectively. This figure is taken from [1].

continues to grow also beyond $N = 25$ and we conclude that (5.73) is the appropriate asymptotic expansion for $G_1(z)$ for large values of z .

This is a remarkable result. To see why, we go back to pure soft-quark diagrams, where no soft gluon effects are considered. This subset of diagrams is described by the coupled iterative integral equations in (3.70) and (3.72) without the exponential Sudakov as part of the integral equations. In this case the integral equations can be solved in closed form and the solutions are given in terms of modified Bessel functions $I_i(2\sqrt{z})$ with $i \in \{0, 1\}$ (see (3.77)). In contrast to the standard Sudakov factor that leads to an exponential suppression in the asymptotic limit, modified Bessel functions are exponentially enhanced in the asymptotic limit

$$I_n(2\sqrt{z}) \approx \frac{1}{2\sqrt{\pi}} \frac{e^{2\sqrt{z}}}{z^{1/4}}, \quad \text{for } z \rightarrow \infty. \quad (5.75)$$

The fact that there is neither an exponential enhancement nor suppression when considering both soft quarks and soft gluons but a *linear* enhancement implies that there is a delicate cancellation between both effects at play. This effect can be further understood by inspecting the differential equation (5.12) which is the starting point for the derivation of the one-fold integral equations that eventually encode the behavior at large z . The second term in (5.12) originates from the exponential in the integrand and is thus connected to soft gluon effects. The third term has this form because of the iterative nature of the integral equations and is, thus, connected to soft-quark effects. If the integral equations were not iterative, i. e. if the integrand were just an exponential, this term would be a constant. The differential equation (5.12) is transformed to the simpler differential equation (5.16) by performing an additional derivative with respect to η and in the result in (5.16) there is no term of the form $\partial_\eta g_1(\rho, \eta)$ because the second and third term in (5.12) have exactly opposite signs. If this cancellation were not exact, the simplified differential equation (5.16) would take a form that would either lead an exponential enhancement or suppression. We also remind the reader that in the final form factor in (3.128) $G_1(z)$ is still multiplied with a global Sudakov such that the result is still overall suppressed exponentially in the asymptotic limit.

We now turn to the asymptotic expansion of $G_{2b}(z)$. Here, the integrated one-fold integral equation has the inhomogeneous term (cf. (5.50))

$$\int_0^\rho d\rho' \mathcal{K}_I(\rho'(\rho - \rho')) C_1(\rho'), \quad (5.76)$$

with the kernel $\mathcal{K}_I(x)$ defined in (5.51). It is convenient to derive a close form of $\mathcal{K}_I(x)$ where all integrals have been performed. Starting from the definition of $\mathcal{K}(x)$ in (5.49) we obtain

$$\mathcal{K}(x) = \int_0^x dx' \frac{x' e^{-x'} - x e^{-x}}{x(x - x')} = \frac{e^{-x} - 1}{x} + e^{-x} (\text{Ei}(x) - \ln(x) - \gamma_E), \quad (5.77)$$

where $\text{Ei}(x)$ is the exponential integral, defined e.g. via the principal value

$$\text{Ei}(x) = -\text{PV} \int_{-x}^\infty dt \frac{e^{-t}}{t}. \quad (5.78)$$

Inserting (5.77) into the definition of $\mathcal{K}_I(x)$ in (5.51) gives

$$\mathcal{K}_I(x) = - \int_0^x dx' \mathcal{K}(x') = -e^{-x} (-\text{Ei}(x) + \ln(x) + \gamma_E). \quad (5.79)$$

In contrast to the exponential suppression in the one-fold integral for $C_1(\rho)$, the kernel $\mathcal{K}_I(x)$ is only suppressed as $1/x + 1/x^2 + \dots$ for large x . This leads to two significant complications compared to the analysis of the asymptotic limit for $G_1(z)$. First, the less severe suppression leads to additional regions since now also regions with $\rho'(\rho - \rho') \gg 1$ are allowed. Second, all regions are now logarithmically divergent. This includes region (A) and (B) since the endpoint of the integrals at infinity is less constraint compared to the exponential suppression. Because only the sum of all regions is finite, the divergences demand that we consistently introduce a regulator across all regions such that single regions are well-defined.

For concreteness, the additional regions are given by

$$\begin{aligned} \text{region (C)} \quad \rho' \sim \rho - \rho' \sim \rho, & \quad \text{such that} \quad \rho'(\rho - \rho') \gg 1, \\ \text{region (D)} \quad \rho' \sim O(1), & \quad \text{such that} \quad \rho'(\rho - \rho') \approx \rho' \rho \gg 1. \end{aligned}$$

Before we turn to any specific region we have to define a regulator and choose an analytic regulator that modifies (5.76) according to

$$\tilde{\mathcal{K}}_{I,\alpha}(\rho^2) = \frac{1}{\rho} \int_0^\rho d\rho' \rho'^\alpha (\rho - \rho')^\alpha \mathcal{K}_I(\rho'(\rho - \rho')) C_1(\rho'), \quad (5.80)$$

where the additional factor of $\rho'^\alpha (\rho - \rho')^\alpha$ is the analytic regulator and we defined $\tilde{\mathcal{K}}_{I,\alpha}(\rho^2)$ for notational convenience. Equivalently, we could choose a regulator where both factors are regulated separately, e. g. $\rho'^\alpha (\rho - \rho')^\beta$. This would have the advantage of additional structure in intermediate expressions. But the expansions in α and β do not commute, which adds an unnecessary layer of complication, and we choose the regulator in (5.80) instead. While we use an analytic regulator across all regions it can nevertheless be useful to separate integration domains at the level of single regions into singular and non-singular parts by means of cutoffs. This allows us to use the analytic regulator only in the singular parts of the integration domain where the integrand can be expanded in the respective limits. We note that working with both analytic regulators and cutoffs at the same time requires to carefully organize the expansions in the cutoff parameter and α in a consistent way

because the expansions do not commute. We illustrate this procedure in the calculation of region (D) below. Up to technicalities related to regulators discussed above, the calculation in regions (A) and (B) proceeds in analogy to the steps leading to (5.63) and (5.64), and we only quote the results. For region (A) we have

$$\tilde{\mathcal{K}}_{I,\alpha}(z) \stackrel{(A)}{=} -\frac{1}{z} \left(\frac{1}{\alpha} - \gamma_E \right) + O(z^{-2}, \alpha), \quad (5.81)$$

where we use $G_1(0) = 1$. Counting powers of ρ^{-1} in the integral definition (5.80) in region (B) we expect that there is a contribution linear in z . But the respective integral vanishes and the expansion starts with a term constant in z

$$\tilde{\mathcal{K}}_{I,\alpha}(z) \stackrel{(B)}{=} -\left(\frac{1}{\alpha} - \gamma_E \right) + \frac{1}{z} \left(\frac{1}{\alpha} - \gamma_E \right) + O(z^{-2}, \alpha), \quad (5.82)$$

where we use the known asymptotic limit for $G_1(z)$ in (5.73). In region (C) the integrand simplifies significantly because the argument of $\mathcal{K}(x)$ becomes large and we have to expand the integration kernel. The occurring integrals can be reduced to Beta-functions and the result is

$$\tilde{\mathcal{K}}_{I,\alpha}(z) \stackrel{(C)}{=} \left(\frac{1}{\alpha} - 1 + \ln(z) \right) - \frac{1}{z} \left(\frac{2}{\alpha} + 2 + 2 \ln(z) \right) + O(z^{-2}, \alpha), \quad (5.83)$$

where we again use the known asymptotic limit for $G_1(z)$ in (5.73). The analysis in region (D) is slightly more involved. The integral becomes

$$\tilde{\mathcal{K}}_{I,\alpha}(z) \stackrel{(D)}{=} \tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; 0, \infty) + O(z^{-2}) = \rho^{-2+\alpha} \int_0^\infty d\rho' \rho'^{-1+\alpha} C_1(\rho') + O(z^{-2}), \quad (5.84)$$

where the second and third argument of $\tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; 0, \infty)$ denote the integration limits. Since ρ' is $O(1)$ in region (D) we cannot use the known analytic form of the asymptotic limit of $C_1(\rho)$ and have to resort to numerical integration over the series expansion using a partial sum with sufficiently high N (cf. (5.74)). We expect the integral (5.84) to develop divergences at both endpoints, which requires to introduce a cutoff λ_0 close to $z = 0$ and a cutoff λ_∞ close to $z = \infty$. Consequently, we split the integration domain according to

$$\tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; 0, \infty) = \tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; 0, \lambda_0) + \tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; \lambda_0, \lambda_\infty) + \tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; \lambda_\infty, \infty). \quad (5.85)$$

Close to $z = 0$ it is sufficient to approximate $C_1(\rho') \approx 1$ to capture all divergences and the integral becomes

$$\begin{aligned} \tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; 0, \lambda_0) &\approx \rho^{-2+\alpha} \int_0^{\lambda_0} d\rho' \rho'^{-1+\alpha} \\ &= \frac{1}{z} \left(\frac{1}{\alpha} + \ln(\lambda_0) + \frac{\ln(z)}{2} \right) + O(\alpha). \end{aligned} \quad (5.86)$$

Similarly, we have to expand up to and including the constant term in ρ' to capture all divergences close to $z = \infty$. We can use the known approximation in the asymptotic limit $C_1(\rho') \approx \rho'^2 - 1$ (cf. (5.73)) to write

$$\begin{aligned} \tilde{\mathcal{K}}_{I,\alpha}^{(D)}(z; \lambda_\infty, \infty) &\approx \rho^{-2+\alpha} \int_{\lambda_\infty}^\infty d\rho' \rho'^{-1+\alpha} (\rho'^2 - 1) \\ &= \frac{1}{z} \left(\frac{1}{\alpha} - \frac{\lambda_\infty^2}{2} + \ln(\lambda_\infty) + \frac{\ln(z)}{2} \right) + O(\alpha). \end{aligned} \quad (5.87)$$

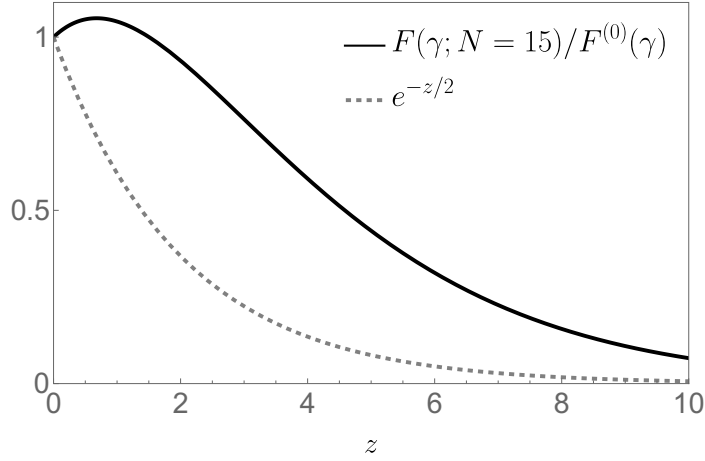


Figure 5.2: Comparison of the large- z fall-off between the standard multiplicative Sudakov factor in the double-logarithmic approximation (dashed curve) and the resummed expression for the physical form factor $F(\gamma)$ (solid curve). Both curves are normalized to the Born expression $F^{(0)}(\gamma)$, and for the solid curve we evaluate the partial sum $F(\gamma; N) = \sum_{n=0}^N \left(\frac{\alpha_s}{4\pi}\right)^n F^{(n)}(\gamma)$ for $N = 15$ with equal quark masses, $\bar{u}_0 = 1/2$. As the plots from Fig. 5.1 show, the black curve is well approximated by this polynomial for $z \leq 10$. We recall that the variable $z = \frac{\alpha_s C_F}{2\pi} L^2$ is related to the large boost γ through $L = \ln(2\gamma)$. This figure is taken from [1].

Collecting the divergent pieces in the limits $\lambda_0 \rightarrow 0$ and $\lambda_\infty \rightarrow \infty$ in (5.86) and (5.87) we define

$$\tilde{\mathcal{K}}_{I,0,\text{finite}}^{(D)}(z; \lambda_0, \lambda_\infty) = \frac{1}{z} \left(\ln(\lambda_0) - \frac{\lambda_\infty^2}{2} + \ln(\lambda_\infty) \right) + \tilde{\mathcal{K}}_0^{(D)}(z; \lambda_0, \lambda_\infty), \quad (5.88)$$

where we set $\alpha = 0$ because there are no divergences in α for finite λ_0 and λ_∞ . We expect that $\tilde{\mathcal{K}}_{I,0,\text{finite}}^{(D)}(z; \lambda_0, \lambda_\infty)$ becomes independent of λ_0 (λ_∞) for sufficiently low λ_0 (high λ_∞). To check the independence of $\lambda_{0/\infty}$ and to calculate the value of $\tilde{\mathcal{K}}_{I,0,\text{finite}}^{(D)}(z; \lambda_0, \lambda_\infty)$ we use the partial sum in (5.74) with $N = 1000$. The result is

$$\tilde{\mathcal{K}}_{I,0,\text{finite}}^{(D)}(z; \lambda_0, \lambda_\infty) = \frac{c - \gamma_E}{z}, \quad (5.89)$$

where c is a constant that we determine by numerical integration. We verified that $c = 1$ up to $O(10^{-4})$ and will use this as an exact value from now on. Summarizing the calculation in region (D) we obtain

$$\tilde{\mathcal{K}}_{I,\alpha}(z) \stackrel{(D)}{=} \frac{1}{z} \left(\frac{2}{\alpha} + 1 - \gamma_E + \ln(z) \right) + O(z^{-2}, \alpha). \quad (5.90)$$

Finally, we can add the contributions from all regions and obtain

$$\tilde{\mathcal{K}}_{I,\alpha}(z) = \ln(z) + \gamma_E - 1 - \frac{1}{z} (\ln(z) + \gamma_E + 1) + O(z^{-2}). \quad (5.91)$$

For the homogeneous term in the one-fold integral equation for $g_{2b}(\rho, \rho)$ in (5.50) we repeat the steps that lead to the differential equation for $G_1(z)$ in (5.65). The result is exactly the same, and we add the expansion of the inhomogeneous term in (5.91). This yields an ordinary differential equation for $1 + G_{2b}(z)$, which can be solved following the same approach as for $G_1(z)$. The solution gives the asymptotic expansion for $1 + G_{2b}(z)$ in the limit of large z

$$1 + G_{2b}(z) = (1 - z) (\ln z + \gamma_E - 2) + O(z^{-1}). \quad (5.92)$$

While the term $z \ln(z)$ is formally leading we remind the reader that it is suppressed by $\sim 1/N_c^2$ in the large N_c limit. In Fig. 5.1 we test the asymptotic expansion of $G_1(z)$ and $G_{2b}(z)$ against partial sums of the known series expansions and find agreement in all tested validity ranges of the partial sums. Furthermore, in Fig. 5.2 we compare our resummed result of the form factor $F(\gamma)$ with an exponential Sudakov factor. For small values of z our result rises before it starts to fall off for larger values of z . Overall the fall off is damped in comparison with the exponential Sudakov factor due to the linear enhancement.

SUMMARY

In this chapter we briefly summarize the findings of the first part of this thesis. To this end, we revisit the steps that lead to the central result of this analysis given by the iterative integral equations in (3.126) and (3.128) that resum the double-logarithmic corrections to the soft-overlap term of $B_c \rightarrow \eta_c$ transition form factors at large hadronic recoil, their independent check using EFT techniques up to the two-loop level and a subsequent analysis of the integral equations.

We started with laying out the method that we used for diagrammatic resummation. In particular, we found that it is sufficient to supply eikonal regions with appropriate *physical* cutoffs to extract the double-logarithmic contribution. At fixed order we identified two kinds of configurations that contribute at the double-logarithmic level: On-shell soft-quark configurations obeying strong-rapidity ordering and on-shell soft-gluon configurations. Extending the fixed-order analysis to all orders we found that only a small subset of all soft-quark diagrams contributes at the double-logarithmic level in the chosen hybrid gauge. The strong-rapidity ordering leads to recursive integral equations with mixing due to the non-trivial iteration of Dirac structures. To obtain a reliable understanding to all orders we analyzed not only the subset of diagrams that we expect to contribute by extrapolating from the fixed-order analysis, but also the remaining set of soft-quark diagrams, which are all eikonalized by the strong-rapidity ordering. A careful analysis employing suitable one-loop building blocks indeed confirmed that only the previously considered subset of soft-quark diagrams contributes at the double-logarithmic level. Soft-gluon corrections factorize and therefore do not lead to new structures in the iterative integral equations. Nevertheless they lead to exponential Sudakov corrections, both as a global factor and as a modification for every rung of the soft-quark ladder, which concluded the derivation of the complete iterative integral equations. These integral equations represent the first successful resummation of logarithms of the soft-overlap term.

Since the derivation of the iterative integral equations involved a number of non-trivial insights, we performed an independent cross-check at fixed order. To this end, we set up a bare factorization formula in SCET. Endpoint-divergent convolution integrals prevent the use of the factorization formula for resummation via RG methods, but it can be used at fixed order provided appropriate regulators are in place. The structure of the factorization formula enabled us to derive pole-cancellation arguments up to the two-loop level. There is only one unknown coefficient at this level. There is a certain freedom to choose from a number of solutions of the pole-cancellation constraints and we picked one that required us to compute the leading ε -pole of the two-loop purely hard-collinear region j_{40} . We employed state-of-the-art methods for loop calculations within an automated setup to determine j_{40} . The result confirmed the iterative integral equations up to this level, which covers many non-trivial aspects of the integral equations.

Finally, we performed a detailed analysis of the integral equations. Although we were unable to solve the integral equations in closed form, we derived on-shell one-fold integral equations that enabled us to formulate iterative relations for the coefficients of the double-logarithmic series. These iteration relations allow us to calculate the double-logarithmic series to, in principle, arbitrary order in an efficient manner. Additionally, we performed a method-of-regions analysis of the one-fold integral equations to determine their behavior in the limit of infinite recoil and found that standard Sudakov suppression is slightly weakened by a linear factor.

Part II

MASSIVE-MASSLESS SUBTRACTION TERMS AT AN ARBITRARY ANGLE

We now come to the second part of this thesis. Instead of trying to understand the logarithmic structure to all orders, here, we are concerned with the IR structure at fixed order. To this end, we calculate the phase-space integral of the double-emission soft limit of generic QCD amplitudes with a massive and a massless emitter at an arbitrary angle to each other. This is a necessary ingredient to extend the nested soft-collinear subtraction scheme [54] to processes with massive partons in the final state. For concreteness, we study the partonic process

$$0 \rightarrow h_1(p_1) + \cdots + h_n(p_n) + H_{n+1}(p_{n+1}) + \cdots + H_N(p_N) + f_1(k_1) + f_2(k_2), \quad (7.1)$$

where h_i are colored on-shell massless partons, $p_i^2 = 0$, $i \in \{1, \dots, n\}$, and H_i are colored on-shell massive partons, $p_i^2 \neq 0$, $i \in \{n+1, \dots, N\}$. Extending our discussion to processes with additional color singlets is straightforward. While h_i and H_i are part of the hard process under consideration, $f_{1/2}$ are additional potentially unresolved, colored, on-shell partons, either two gluons or a quark-anti-quark pair. In particular, we consider the limit in which both k_1 and k_2 become soft simultaneously, or in terms of hierarchies of scalar products

$$k_1 \cdot k_2 \ll p_i \cdot k_j \ll p_k \cdot p_\ell, \quad (7.2)$$

and “simultaneously” refers to the ratio of energies of the unresolved partons being constant. In short, we refer to this hierarchy as the soft limit and write

$$\lim_{k_1, k_2 \rightarrow 0} A, \quad (7.3)$$

where A is some test function and the limit typically does not exist in the strict mathematical sense and instead the leading asymptotics are projected out. It is well-known that the leading asymptotics of an amplitude with soft partons factorizes into eikonal factors and color-correlated matrix elements. We are interested in the soft limit of the tree-level matrix element of the process (7.1) for which we can write [152]:

- if $f_{1,2}$ are gluons,

$$\begin{aligned} \lim_{k_1, k_2 \rightarrow 0} |\mathcal{M}^{gg}(\{p\}, k_1, k_2)|^2 \approx g_{s,b}^4 \left\{ \frac{1}{2} \sum_{i,j,k,l}^N \mathcal{S}_{ij}(k_1) \mathcal{S}_{kl}(k_2) |\mathcal{M}^{\{(ij),(kl)\}}(\{p\})|^2 \right. \\ \left. - C_A \sum_{i,j}^N \mathcal{S}_{ij}(k_1, k_2) |\mathcal{M}^{(ij)}(\{p\})|^2 \right\}, \end{aligned} \quad (7.4)$$

- if $f_1 = q$ and $f_2 = \bar{q}$,

$$\lim_{k_1, k_2 \rightarrow 0} |\mathcal{M}^{q\bar{q}}(\{p\}, k_1, k_2)|^2 \approx g_{s,b}^4 T_R \sum_{i,j}^N \mathcal{I}_{ij}(k_1, k_2) |\mathcal{M}^{(ij)}(\{p\})|^2. \quad (7.5)$$

The color-correlated matrix elements are defined as

$$\begin{aligned} |\mathcal{M}^{\{(ij),(kl)\}}(\{p\})|^2 &= \langle \mathcal{M}(\{p\}) | \{T_i \cdot T_j, T_k \cdot T_l\} | \mathcal{M}(\{p\}) \rangle, \\ |\mathcal{M}^{\{(ij)\}}(\{p\})|^2 &= \langle \mathcal{M}(\{p\}) | T_i \cdot T_j | \mathcal{M}(\{p\}) \rangle, \end{aligned} \quad (7.6)$$

where T_i are color operators introduced in [153] and $\{\cdot, \cdot\}$ is an anti-commutator. We give more details on this notation in Appendix C. The Abelian part of the eikonal function for two gluon emissions factorizes into two single gluon emission eikonal functions $\mathcal{S}_{ij}(k)$

$$\mathcal{S}_{ij}(k) = \frac{(p_i \cdot p_j)}{(p_i \cdot k)(p_j \cdot k)}. \quad (7.7)$$

The non-Abelian part originates from splittings of gluons and, therefore, the dependence on k_1 and k_2 does not factorize. We can write [114, 152, 154]

$$\mathcal{S}_{ij}(k_1, k_2) = \mathcal{S}_{ij}^0(k_1, k_2) + [m_i^2 \mathcal{S}_{ij}^m(k_1, k_2) + m_j^2 \mathcal{S}_{ji}^m(k_1, k_2)], \quad (7.8)$$

where $\mathcal{S}_{ij}^0(k_1, k_2)$ is given by

$$\begin{aligned} \mathcal{S}_{ij}^0(k_1, k_2) &= \frac{(1-\varepsilon)}{(k_1 \cdot k_2)^2} \frac{[(p_i \cdot k_1)(p_j \cdot k_2) + i \leftrightarrow j]}{(p_i \cdot k_{12})(p_j \cdot k_{12})} \\ &\quad - \frac{(p_i \cdot p_j)^2}{2(p_i \cdot k_1)(p_j \cdot k_2)(p_i \cdot k_2)(p_j \cdot k_1)} \left[2 - \frac{[(p_i \cdot k_1)(p_j \cdot k_2) + i \leftrightarrow j]}{(p_i \cdot k_{12})(p_j \cdot k_{12})} \right] \\ &\quad + \frac{(p_i \cdot p_j)}{2(k_1 \cdot k_2)} \left[\frac{2}{(p_i \cdot k_1)(p_j \cdot k_2)} + \frac{2}{(p_j \cdot k_1)(p_i \cdot k_2)} - \frac{1}{(p_i \cdot k_{12})(p_j \cdot k_{12})} \right. \\ &\quad \left. \times \left(4 + \frac{[(p_i \cdot k_1)(p_j \cdot k_2) + i \leftrightarrow j]^2}{(p_i \cdot k_1)(p_j \cdot k_2)(p_i \cdot k_2)(p_j \cdot k_1)} \right) \right], \end{aligned} \quad (7.9)$$

and the contributions specific to massive partons are

$$\begin{aligned} \mathcal{S}_{ij}^m(k_1, k_2) &= \frac{(p_i \cdot p_j)(p_j \cdot k_{12})}{2(p_i \cdot k_1)(p_j \cdot k_2)(p_i \cdot k_2)(p_j \cdot k_1)(p_i \cdot k_{12})} \\ &\quad - \frac{1}{2(k_1 \cdot k_2)(p_i \cdot k_{12})(p_j \cdot k_{12})} \left(\frac{(p_j \cdot k_1)^2}{(p_i \cdot k_1)(p_j \cdot k_2)} + \frac{(p_j \cdot k_2)^2}{(p_i \cdot k_2)(p_j \cdot k_1)} \right). \end{aligned} \quad (7.10)$$

For the quark-anti-quark pair the eikonal function is

$$\mathcal{I}_{ij}(k_1, k_2) = \frac{[(p_i \cdot k_1)(p_j \cdot k_2) + i \leftrightarrow j] - (p_i \cdot p_j)(k_1 \cdot k_2)}{(k_1 \cdot k_2)^2 (p_i \cdot k_{12})(p_j \cdot k_{12})}. \quad (7.11)$$

In an EFT language the hierarchy (7.2) requires to introduce hard-collinear fields for the massless partons and HQET fields for the massive partons. The soft currents then correspond to soft matrix elements that are generated by decoupling transformations [23] for the hard-collinear and HQET fields. We refer, for instance, to [155, 156] for a more detailed discussion.

We can use color conservation

$$\sum_{i=1}^N T_i | \mathcal{M}(\{p\}) \rangle = 0, \quad (7.12)$$

and symmetry under the exchange of p_i and p_j , $\mathcal{S}_{ij} = \mathcal{S}_{ji}$, $\mathcal{I}_{ij} = \mathcal{I}_{ji}$, to rearrange the sums in (7.4) and (7.5)

$$\sum_{i,j}^N \mathcal{S}_{ij}(k_1, k_2) |\mathcal{M}^{(ij)}(\{p\})|^2 = \sum_{i < j}^N \tilde{\mathcal{S}}_{ij}(k_1, k_2) |\mathcal{M}^{(ij)}(\{p\})|^2, \quad (7.13)$$

$$\sum_{i,j}^N \mathcal{I}_{ij}(k_1, k_2) |\mathcal{M}^{(ij)}(\{p\})|^2 = \sum_{i < j}^N \tilde{\mathcal{I}}_{ij}(k_1, k_2) |\mathcal{M}^{(ij)}(\{p\})|^2, \quad (7.14)$$

where $\tilde{\mathcal{S}}_{ij}(k_1, k_2)$ and $\tilde{\mathcal{I}}_{ij}(k_1, k_2)$ are

$$\tilde{\mathcal{S}}_{ij} = 2\mathcal{S}_{ij} - \mathcal{S}_{ii} - \mathcal{S}_{jj}, \quad (7.15)$$

$$\tilde{\mathcal{I}}_{ij} = 2\mathcal{I}_{ij} - \mathcal{I}_{ii} - \mathcal{I}_{jj}, \quad (7.16)$$

which has the advantage that divergences are physical and potential cancellations are taken into account. In analogy to (1.91) we define the phase space integrals of the double-emission eikonal functions as [54]

$$\mathfrak{S} [\mathcal{S}_{ij} \mathcal{S}_{kl}] = \int [dk_m] [dk_n] \theta(E_{\max} - k_m^0) \theta(k_m^0 - k_n^0) \mathcal{S}_{ij}(k_m) \mathcal{S}_{kl}(k_n), \quad (7.17)$$

$$\mathfrak{S} [\Xi_{ij}] = \int [dk_m] [dk_n] \theta(E_{\max} - k_m^0) \theta(k_m^0 - k_n^0) \Xi_{ij}(k_m, k_n), \quad (7.18)$$

where Ξ_{ij} represents either $\tilde{\mathcal{S}}_{ij}$ or $\tilde{\mathcal{I}}_{ij}$, $[dk_{m/n}]$ are defined analogously to (1.92) and the second Heaviside function imposes an energy ordering among k_m and k_n . Inserting the identity

$$\theta(b - a) = \int_0^1 dx \delta(xb - a) b, \quad (7.19)$$

in (7.18) yields

$$\mathfrak{S} [\Xi_{ij}] = \int_0^1 \frac{dx}{x} \int [dk_m] [dk_n] \delta\left(1 - \frac{k_m^0}{xE_{\max}}\right) \theta(k_m^0 - k_n^0) \Xi_{ij}(k_m, k_n). \quad (7.20)$$

We observe that Ξ_{ij} is homogeneous under simultaneous rescaling of k_m and k_n

$$\Xi_{ij}(\omega k_m, \omega k_n) = \frac{\Xi_{ij}(k_m, k_n)}{\omega^4}. \quad (7.21)$$

The homogeneity is a consequence of the way the double soft limit is defined and the power of 4 is the degree of divergence of two soft emissions. We can effectively factorize the energy integration of $[dk_m]$ from the remaining integral by using (7.21) with $\omega = xE_{\max}$ to rescale the momenta in (7.20)

$$\begin{aligned} \mathfrak{S} [\Xi_{ij}] &= \int_0^1 \frac{dx}{x} \frac{(xE_{\max})^{2(d-2)}}{(xE_{\max})^4} \int [dl_m] [dl_n] \delta(1 - l_m \cdot P) \theta(l_m \cdot P - l_n \cdot P) \Xi_{ij}(l_m, l_n) \\ &= -\frac{1}{4\epsilon E_{\max}^{4\epsilon}} \int [dl_m] [dl_n] \delta(1 - l_m \cdot P) \theta(l_m \cdot P - l_n \cdot P) \Xi_{ij}(l_m, l_n), \end{aligned} \quad (7.22)$$

where $l_{\text{m}/\text{n}} = k_{\text{m}/\text{n}}/(xE_{\text{max}})$ and $P = (1, \vec{0})$. Since the single emission eikonal functions $\mathcal{S}_{ij}(k_{\text{m}/\text{n}})$ in $\mathfrak{S}[\mathcal{S}_{ij}\mathcal{S}_{kl}]$ are homogeneous under separate rescalings of k_{m} and k_{n} , we can factorize both energy integrations in this case and obtain

$$\mathfrak{S}[\mathcal{S}_{ij}\mathcal{S}_{kl}] = \frac{1}{8\varepsilon^2 E_{\text{max}}^{4\varepsilon}} \int [dl_{\text{m}}] \delta(1 - l_{\text{m}} \cdot P) \mathcal{S}_{ij}(l_{\text{m}}) \int [dl_{\text{n}}] \delta(1 - l_{\text{n}} \cdot P) \mathcal{S}_{kl}(l_{\text{n}}). \quad (7.23)$$

The remaining single emission integrals are well-known [157, 158], but we will repeat their calculation as an illustrative example before turning to $\mathfrak{S}[\Xi_{ij}]$. Eventually, we are interested in $\mathfrak{S}[\Xi_{ij}]$ where one of the external momenta is massive and the other is massless with an arbitrary angle between them. Without loss of generality we can choose p_i to be massive ($p_i^2 = m^2$) and p_j to be massless ($p_j^2 = 0$). It is then convenient to parameterize $p_{i/j}$ as

$$p_i = E_i(1, \beta \vec{n}_i), \quad p_j = E_j(1, \vec{n}_j), \quad (7.24)$$

where

$$\beta = \sqrt{1 - \frac{m^2}{E_i^2}}. \quad (7.25)$$

The scalar products become

$$p_i^2 = m^2, \quad P^2 = 1, \quad p_i \cdot p_j = E_i E_j (1 - \beta \cos \theta), \quad p_{i/j} \cdot P = E_{i/j}, \quad (7.26)$$

where $\cos \theta = \vec{n}_i \cdot \vec{n}_j$. Inserting these definitions into (7.8), (7.9), (7.10) and (7.11) the dependence on E_j drops out and only an implicit dependence on E_i through β remains. Since the explicit dependence on $E_{i/j}$ drops out, $\mathfrak{S}[\Xi_{ij}]$ has a non-trivial dependence on only two parameters, β and $\cos \theta$. Having set up all relevant definitions we continue with the calculation of the single emission current before turning to $\mathfrak{S}[\Xi_{ij}]$.

As we mentioned previously, the integrals of single emission eikonal functions are known in the literature [157, 158]. We nevertheless discuss its calculation in some detail as a concrete example of employing the differential equations technique (cf. Sec. 1.4). Also for the single emission current we consider the case with one massless and one massive external momentum and use the definitions for momenta $p_{i/j}$ from the last section. The integral we aim to calculate can be taken from (7.23)

$$\begin{aligned}\mathcal{G}[\mathcal{S}_{ij}] &= \int [dl_m] \delta(1 - l_m \cdot P) \mathcal{S}_{ij}(l_m) \\ &= \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{1 - \beta \cos \theta}{(1 - \beta \vec{n}_i \cdot \vec{n}_m)(1 - \vec{n}_j \cdot \vec{n}_m)},\end{aligned}\tag{8.1}$$

where we used the delta function to integrate over the radial component of the phase space integral and $d\Omega_m^{(d-1)}$ is the remaining angular integral measure of a $d - 1$ dimensional sphere represented by $\vec{n}_m$. The integral (8.1) can be solved by direct calculation to all orders in ϵ , the result being an Appell function [159]. Instead, here we intend to calculate it by means of the differential equation technique. Since this requires an IBP reduction (see Sec. 1.3 and 1.4), it is convenient to introduce the notation

$$I_{a_1 a_2} \equiv \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{(1 - \beta \cos \theta)}{(1 - \beta \vec{n}_i \cdot \vec{n}_m)^{a_1} (1 - \vec{n}_j \cdot \vec{n}_m)^{a_2}}, \quad a_i \in \mathbb{Z},\tag{8.2}$$

and to rewrite the angular integral in terms of a loop integral and delta functions

$$\frac{d\Omega_m^{(d-1)}}{2} = d^d l_m \delta^+(l_m^2) \delta(1 - l_m \cdot P).\tag{8.3}$$

We then use reverse unitarity [160]

$$2\pi i \delta(p^2 - m^2) \rightarrow \frac{1}{p^2 - m^2 + i0^+} - \frac{1}{p^2 - m^2 - i0^+},\tag{8.4}$$

to bring the delta functions into a form that can be handled in an IBP reduction. Although the integrals $I_{a_1 a_2}$ depend on two parameters, β and $\cos \theta$, we only use a differential equation with respect to β . In principle, one then has to determine boundary conditions including the functional dependence on $\cos \theta$. However, using $\beta = 0$ as a boundary, the boundary conditions become independent of $\cos \theta$. Introducing the shorthand $y = \cos \theta$ and performing the derivative with respect to β , we obtain

$$\partial_\beta \vec{I}(d, \beta) = M(d, \beta) \vec{I}(d, \beta),\tag{8.5}$$

where

$$M = \begin{pmatrix} -\frac{y}{1-\beta y} & 0 & 0 \\ \frac{1-2\epsilon}{\beta(1-\beta^2)} & -\frac{1-\beta^2-2\epsilon(1-\beta y)}{\beta(1-\beta^2)(1-\beta y)} & 0 \\ \frac{1-2\epsilon}{\beta(1-\beta^2)(1-\beta y)} & \frac{2\beta\epsilon}{(1-\beta^2)(1-\beta y)} & \frac{2\epsilon}{\beta(1-\beta y)} \end{pmatrix},\tag{8.6}$$

and $\vec{I}(d, \beta) = (I_{00}, I_{10}, I_{11})$ are the MIs. Not coincidentally, (8.6) is identical with (1.76) and we can, therefore, adopt the discussion from Sec. 1.4. With the help of (D.61) it is straightforward to obtain the boundary conditions

$$\vec{I}(d, \beta = 0) = \mathcal{N}_\varepsilon \left(1, 1, -\frac{1}{2} \frac{(1-2\varepsilon)}{\varepsilon} \right), \quad (8.7)$$

where

$$\mathcal{N}_\varepsilon = \frac{\pi^\varepsilon}{4\pi^2} \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}. \quad (8.8)$$

Following the procedure laid out in Sec. 1.4, we transform the boundary conditions to the canonical basis, solve the differential equation in the canonical basis in terms of MPLs and transform the result back to the original basis. Defining

$$I_{11} = \frac{(1-2\varepsilon)}{\varepsilon} \mathcal{N}_\varepsilon \sum_{n=0} I_{11}^{(n)} \varepsilon^n, \quad (8.9)$$

the result for I_{11} up to $n = 2$ is given by

$$\begin{aligned} I_{11}^{(0)} &= -\frac{1}{2}, \\ I_{11}^{(1)} &= G\left(\frac{1}{y}; \beta\right) - \frac{1}{2}G(-1; \beta) - \frac{1}{2}G(1; \beta), \\ I_{11}^{(2)} &= \frac{1}{2} \left[4G\left(0, \frac{1}{y}; \beta\right) + 2G\left(\frac{1}{y}, -1; \beta\right) + 2G\left(\frac{1}{y}, 1; \beta\right) - 4G\left(\frac{1}{y}, \frac{1}{y}; \beta\right) \right. \\ &\quad \left. + G(-1, -1; \beta) - G(-1, 1; \beta) - 2G(0, -1; \beta) - 2G(0, 1; \beta) - G(1, -1; \beta) + G(1, 1; \beta) \right]. \end{aligned} \quad (8.10)$$

Using the identities

$$\begin{aligned} G(\vec{a}_n; z) &= \frac{1}{n!} \ln^n \left(1 - \frac{z}{a} \right), \\ G(\vec{0}_{n-1}, a; z) &= -\text{Li}_n \left(\frac{z}{a} \right), \end{aligned} \quad (8.11)$$

where $\vec{a}_n = (a, \dots, a)$ is a n -dimensional vector indicating n identical arguments, Li_n are polylogarithms, and for $a \neq b$ different from zero

$$G(a, b; z) = \text{Li}_2 \left(\frac{b-z}{b-a} \right) - \text{Li}_2 \left(\frac{b}{b-a} \right) + \ln \left(1 - \frac{z}{b} \right) \ln \left(\frac{z-a}{b-a} \right), \quad (8.12)$$

we can rewrite the MPLs in terms of (di)logarithms. Simplifying the resulting expressions, we obtain

$$\begin{aligned} I_{11}^{(0)} &= -\frac{1}{2}, \\ I_{11}^{(1)} &= \ln \left(\frac{1 - \beta \cos \theta}{\sqrt{1 - \beta^2}} \right), \\ I_{11}^{(2)} &= - \left[\frac{1}{4} \ln^2 \left(\frac{1 - \beta}{1 + \beta} \right) + \ln \left(\frac{1 - \beta \cos \theta}{1 + \beta} \right) \ln \left(\frac{1 - \beta \cos \theta}{1 - \beta} \right) \right. \\ &\quad \left. + \text{Li}_2 \left(1 - \frac{1 - \beta \cos \theta}{1 + \beta} \right) + \text{Li}_2 \left(1 - \frac{1 - \beta \cos \theta}{1 - \beta} \right) \right], \end{aligned} \quad (8.13)$$

which matches with the result in the literature [157, 158]. Having discussed the single emission integral, we now turn to the calculation of $\mathfrak{S} [\Xi_{ij}]$.

There are multiple possible strategies to calculate $\mathcal{S}[\Xi_{ij}]$. In contrast to $\mathcal{G}[\mathcal{S}_{ij}]$ (cf. (8.1)), $\mathcal{S}[\Xi_{ij}]$ contains a Heaviside function implying that it is not possible to employ reverse unitarity in a standard way. There are two ways to circumvent this specific problem. The first one is the strategy applied in the calculation of the massless-massless subtraction terms with an arbitrary angle [55] and the massive-massive subtractions in back-to-back kinematics [56]. In that approach, one treats the Heaviside function for a delta function with the help of (7.19). However, since the dependence on the energy does not factorize in this case, there is an additional parameter present in the resulting integrals that can be calculated by means of the differential equation method. An additional complication of this strategy is that the remaining energy integration is divergent and one has to determine the full ε -dependence of the divergent pieces with a method of regions analysis. We employ a similar strategy to calculate an expansion of $\mathcal{S}[\Xi_{ij}]$ around $\beta = 0$. An alternative strategy to deal with the Heaviside function is to incorporate it directly in the IBP reduction [161]. This has the advantage that there is no additional integral required and the integrals to be calculated via the differential equation method depend only on two parameters, β and $\cos \theta$. We used this strategy in [2] to obtain the analytic result for $\mathcal{S}[\Xi_{ij}]$ with full dependence on β and up to $O(\varepsilon^0)$, but we will not provide further details here.

There is yet another approach that is based on a more physical strategy. It does not rely on IBPs and the differential equation method, and therefore does not suffer from the non-factorizing energy integral. While the original definition of $\mathcal{S}[\Xi_{ij}]$ in (7.18) is frame-dependent due to the arguments of the Heaviside functions, the rewritten form in (7.22) is Lorentz-invariant because the energy components were replaced by scalar products with the auxiliary vector P . This allows for a transformation to the rest frame of the massive external momenta which simplifies the calculation because there is no angle between the external momenta $p_{i/j}$ in this frame. The drawback is that the integration domain imposed by the Heaviside function becomes more complicated. We used this strategy in [2] to obtain the analytic result for $\mathcal{S}[\Xi_{ij}]$ with full dependence on β up to $O(\varepsilon^{-1})$ and numerically with full dependence on β up to $O(\varepsilon^0)$, but we will not provide further details here.

In the following, we present the computation of the β -expansion of $\mathcal{S}[\Xi_{ij}]$ around $\beta = 0$. To this end, we can exploit the fact that in the limit $\beta \rightarrow 0$ there is no other singular region of the angular integrals. In the language of the method of regions this means that there is only the hard region which implies that an expansion in β without the need to consider multiple regions is possible. The first step is to employ (7.19) to write also the second Heaviside function as a delta function

$$\mathcal{S}[\Xi_{ij}] = -\frac{1}{4\varepsilon E_{\max}^{4\varepsilon}} \int_0^1 dz \mathcal{K}[\Xi_{ij}](z), \quad (9.1)$$

where

$$\mathcal{K}[\Xi_{ij}](z) = \int [dl_m][dl_n] \delta(1 - l_m \cdot P) \delta(z - l_n \cdot P) \Xi_{ij}(l_m, l_n). \quad (9.2)$$

This allows us to use reverse unitarity to perform an IBP reduction of all occurring integrals. All integrals can be written as¹

$$\Delta_{abc}^{def}(g) = \int \frac{[dl_m][dl_n]\delta(1-l_m \cdot P)\delta(z-l_n \cdot P)}{(l_m \cdot p_i)^a(l_n \cdot p_i)^b(l_{mn} \cdot p_i)^c(l_m \cdot p_j)^d(l_n \cdot p_j)^e(l_{mn} \cdot p_j)^f(l_m \cdot l_n)^g}, \quad (9.3)$$

where $l_{mn} = l_m + l_n$. We use FIRE6 [86] for IBP reductions. In total there are 59 MIs, which are given in Appendix D.1. We write the expansion in β as

$$\mathcal{K}[\Xi_{ij}](z) = \sum_{a=0}^{\infty} \beta^a \mathcal{K}^{(a)}[\Xi_{ij}](z), \quad (9.4)$$

where $\mathcal{K}^{(a)}[\Xi_{ij}](z)$ are independent of β , and we aim to calculate all terms up to $a = 4$. We emphasize that this expansion starts with a term constant in β . Nevertheless, the reduction produces spurious poles up to β^{-3} and we have to determine some MIs up to β^7 in order to reach $a = 4$. The cancellation of spurious poles can later be used as a consistency check on intermediate results.

We derive the differential equation matrix for all MIs

$$\partial_\beta \vec{N}_0(\beta) = M_0(\beta) \vec{N}_0(\beta), \quad (9.5)$$

where $\vec{N}_0(\beta)$ is the vector of all MIs and we notice that $M_0(\beta)$ has poles in β up to β^{-3} . These poles are again spurious, since they would lead to singularities that are not expected in the MIs. Having poles up to β^{-3} is inconvenient and we, therefore, transform to a basis that has at most β^{-1} poles. The differential equation then takes the form

$$\partial_\beta \vec{N}(\beta) = M(\beta) \vec{N}(\beta), \quad (9.6)$$

and the expansion of $M(\beta)$ and $\vec{N}(\beta)$ in β is

$$\vec{N}(\beta) = \sum_{j=0}^{\infty} \beta^j \vec{N}^{(j)}, \quad M(\beta) = \sum_{i=-1}^{\infty} \beta^i M^{(i)}. \quad (9.7)$$

Plugging (9.7) into (9.6) it is straightforward to derive

$$\vec{N}^{(j+1)} = \frac{1}{j+1} \sum_{k=0}^{j+1} M^{(j-k)} \vec{N}^{(k)}, \quad j \in \mathbb{N} \quad (9.8)$$

which can be used recursively to determine $\vec{N}^{(j+1)}$. We note that higher β -poles in $M(\beta)$ would lead to higher limits of the sum in (9.8). For instance, poles of $M(\beta)$ up to β^{-2} would imply that the upper limit of the sum is $j+2$. Then, $j=0$ can be used to determine $\vec{N}^{(2)}$ rather than $\vec{N}^{(1)}$ and, hence, $\vec{N}^{(1)}$ would have to be determined by other means.

The starting point to solve (9.8) recursively is $\vec{N}^{(0)}$, the boundary in the transformed basis at $\beta = 0$. In the original basis the boundary is found by performing a separate IBP reduction with $\beta = 0$. The result is that there are 3 MIs, that depend on z only, since there is no angle between p_i and p_j at $\beta = 0$

$$\Delta_{0,0,0}^{0,0,0}(0) = \int [dl_m][dl_n]\delta(1-l_m \cdot P)\delta(z-l_n \cdot P), \quad (9.9)$$

¹ Note that this definition is different from (4.3) in [2], where z is set to 1 in the second delta function.

$$\Delta_{0,0,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_{mn} \cdot p_j)}, \quad (9.10)$$

$$\Delta_{0,0,0}^{0,0,2}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_{mn} \cdot p_j)^2}, \quad (9.11)$$

which are identical with J_{51} , J_{29} , J_{38} in Appendix D.1, respectively. We give some details on their calculation with exact dependence on ε in Appendix D.2. The transformation to the basis with only single β -poles in $M(\beta)$ has itself poles in β up to β^{-2} . Thus, we have to determine higher β -terms of some MIs to fix the boundary in the transformed system. Here, we leverage that the poles in $M_0(\beta)$ in (9.5), and in higher powers $M_0^p(\beta)$, $p > 1$, representing higher derivatives, must be spurious. These conditions are sufficient to express the required higher orders in β of $\vec{N}_0$ in terms of the MIs for $\beta = 0$ in (9.9) and, finally, to obtain $\vec{N}^{(0)}$. We use FeynCalc's [162, 163] interface to FERMAT [164] to solve (9.8) iteratively. Transforming back to the original basis yields all MIs to the required order in β . Finally, we can plug the MIs into the reduced $\mathcal{K}[\Xi_{ij}](z)$ to obtain all terms up to $a = 4$ in (9.4). The spurious β -poles in $\mathcal{K}[\Xi_{ij}](z)$ vanish as expected.

The last step to obtain $\mathfrak{S}[\Xi_{ij}]$ is to integrate over z according to (9.1). The dependence of $\mathcal{K}[\Xi_{ij}](z)$ on β and z is fully separated and, thus, we can integrate over z order-by-order in β . For $\Xi_{ij} = \tilde{S}_{ij}$ the z -integral is divergent in the limit $z \rightarrow 0$. Physically this corresponds to a large hierarchy in the soft limit of the two potentially unresolved gluons. There are no such divergences for $\Xi_{ij} = \tilde{I}_{ij}$, since single soft (anti)quarks are power suppressed. Since we did not expand in ε up to this point, the divergences of the z -integral in the limit $z \rightarrow 0$ are regulated dimensionally. For terms that do not yield divergences in this limit it is more convenient to expand in ε prior to integrating over z . Defining

$$\mathfrak{S}[\Xi_{ij}] = \sum_{a=0}^{\infty} \beta^a \mathfrak{S}^{(a)}[\Xi_{ij}], \quad (9.12)$$

allows us to give the result order-by-order in β . For $\mathfrak{S}[\tilde{S}_{ij}]$ we obtain

$$\begin{aligned} \mathfrak{S}^{(0)}[\tilde{S}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[-\frac{1}{4\varepsilon^3} + \frac{1}{24\varepsilon^2} + \frac{1}{\varepsilon} \left(-\frac{\pi^2}{24} + \frac{47}{36} - \frac{11}{6} \ln(2) \right) \right. \\ & + \left(-\frac{11\zeta(3)}{8} + \frac{\pi^2}{12} - \frac{553}{108} + \frac{11 \ln^2(2)}{6} + \frac{269 \ln(2)}{36} \right) \\ & + \varepsilon \left(-2\text{Li}_4\left(\frac{1}{2}\right) + \frac{99\zeta(3)}{8} - \frac{7}{4}\zeta(3) \ln(2) - \frac{47\pi^2}{144} + \frac{3887}{324} - \frac{29\pi^4}{1440} - \frac{1}{12} \ln^4(2) \right. \\ & \left. \left. - \frac{11 \ln^3(2)}{9} + \frac{1}{12} \pi^2 \ln^2(2) - \frac{341 \ln^2(2)}{36} + \frac{11}{36} \pi^2 \ln(2) - \frac{1133 \ln(2)}{54} \right) + \mathcal{O}(\varepsilon^2) \right], \quad (9.13) \end{aligned}$$

$$\begin{aligned} \mathfrak{S}^{(1)}[\tilde{S}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[-\frac{y}{\varepsilon^2} - \frac{11y}{6\varepsilon} + \left(-\frac{\pi^2 y}{6} + \frac{23y}{9} - \frac{22y}{3} \ln(2) \right) \right. \\ & \left. + \varepsilon \left(-\frac{11y\zeta(3)}{2} - \pi^2 y - \frac{119y}{54} + \frac{4}{3} y \ln^2(2) + \frac{137}{9} y \ln(2) \right) + \mathcal{O}(\varepsilon^2) \right], \quad (9.14) \end{aligned}$$

$$\begin{aligned} \mathfrak{S}^{(2)}[\tilde{S}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[\frac{1}{\varepsilon^2} \left(\frac{1}{2} - \frac{y^2}{2} \right) + \frac{1}{\varepsilon} \left(\frac{5}{4} - \frac{23y^2}{12} \right) + \left(-\frac{\pi^2 y^2}{12} - \frac{61y^2}{18} - \frac{11}{3} y^2 \ln(2) \right) \right. \\ & \left. + \frac{\pi^2}{12} + \frac{5}{6} + \frac{11 \ln(2)}{3} \right) + \varepsilon \left(-\frac{11y^2\zeta(3)}{4} - \frac{7\pi^2 y^2}{6} - \frac{173y^2}{108} - \frac{7}{3} y^2 \ln^2(2) \right) \end{aligned}$$

$$+\frac{28}{9}y^2\ln(2)+\frac{11\zeta(3)}{4}+\frac{19\pi^2}{18}+\frac{37}{108}+\frac{7\ln^2(2)}{3}-\frac{70\ln(2)}{9}\Big)+O(\varepsilon^2)\Big], \quad (9.15)$$

$$\begin{aligned} \mathfrak{S}^{(3)}[\widetilde{S}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[-\frac{y^3}{3\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{2y}{3} - \frac{31y^3}{18} \right) + \left(-\frac{\pi^2 y^3}{18} - \frac{265y^3}{54} - \frac{22}{9}y^3\ln(2) + \frac{79y}{18} \right) \right. \\ & + \varepsilon \left(-\frac{17y^3}{24} - \frac{46\pi^2 y^3}{27} + \frac{11\pi^2 y}{9} - \frac{133y}{216} - \frac{8}{3}y\ln(2) \right. \\ & \left. \left. - \frac{11y^3\zeta(3)}{6} - \frac{20}{3}y^3\ln^2(2) + \frac{137}{27}y^3\ln(2) + \frac{28}{3}y\ln^2(2) \right) + O(\varepsilon^2) \right], \quad (9.16) \end{aligned}$$

$$\begin{aligned} \mathfrak{S}^{(4)}[\widetilde{S}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[\frac{1}{\varepsilon^2} \left(\frac{1}{4} - \frac{y^4}{4} \right) + \frac{1}{\varepsilon} \left(-\frac{37y^4}{24} + \frac{y^2}{2} + \frac{77}{120} \right) \right. \\ & + \left(-\frac{\pi^2 y^4}{24} - \frac{433y^4}{80} - \frac{11}{6}y^4\ln(2) + \frac{451y^2}{120} + \frac{\pi^2}{24} - \frac{709}{3600} + \frac{11\ln(2)}{6} \right) \\ & + \varepsilon \left(-\frac{211}{72}\pi^2 y^4 + \frac{4203y^4}{800} + \frac{35\pi^2 y^2}{12} - \frac{35603y^2}{3600} - \frac{7\pi^2}{72} + \frac{100967}{36000} \right. \\ & + \frac{11171}{720}y^4\ln(2) + 17y^2\ln^2(2) - \frac{2293}{120}y^2\ln(2) - \frac{13\ln^2(2)}{5} + \frac{3919\ln(2)}{3600} \\ & \left. \left. - \frac{11y^4\zeta(3)}{8} - \frac{44}{3}y^4\ln^2(2) + \frac{11\zeta(3)}{8} \right) + O(\varepsilon^2) \right], \quad (9.17) \end{aligned}$$

where again $y = \cos \theta$ and ζ is the Riemann ζ function. And similarly for $\mathfrak{S}[\widetilde{I}_{ij}]$

$$\begin{aligned} \mathfrak{S}^{(0)}[\widetilde{I}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[-\frac{1}{12\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{25}{72} - \frac{\ln(2)}{3} \right) \right. \\ & + \left(-\frac{275}{216} + \frac{\ln^2(2)}{3} + \frac{59\ln(2)}{36} \right) + \varepsilon \left(\frac{7\zeta(3)}{4} - \frac{23\pi^2}{144} + \frac{2737}{648} - \frac{2}{9}\ln^3(2) \right. \\ & \left. \left. - \frac{95\ln^2(2)}{36} + \frac{1}{18}\pi^2\ln(2) - \frac{577\ln(2)}{108} \right) + O(\varepsilon^2) \right], \quad (9.18) \end{aligned}$$

$$\begin{aligned} \mathfrak{S}^{(1)}[\widetilde{I}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[-\frac{y}{3\varepsilon} + \left(\frac{13y}{18} - \frac{4y}{3}\ln(2) \right) \right. \\ & \left. + \varepsilon \left(-\frac{\pi^2 y}{3} + \frac{14y}{27} - \frac{2}{3}y\ln^2(2) + \frac{35}{9}y\ln(2) \right) + O(\varepsilon^2) \right], \quad (9.19) \end{aligned}$$

$$\begin{aligned} \mathfrak{S}^{(2)}[\widetilde{I}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[\frac{1}{\varepsilon} \left(\frac{1}{6} - \frac{y^2}{6} \right) + \left(\frac{y^2}{36} - \frac{2}{3}y^2\ln(2) - \frac{1}{4} + \frac{2\ln(2)}{3} \right) + \varepsilon \left(-\frac{\pi^2 y^2}{2} + \frac{349y^2}{108} \right. \right. \\ & \left. \left. - \frac{7}{3}y^2\ln^2(2) + \frac{47}{18}y^2\ln(2) + \frac{7\pi^2}{18} - \frac{199}{108} + \frac{5\ln^2(2)}{3} - \frac{19\ln(2)}{6} \right) + O(\varepsilon^2) \right], \quad (9.20) \end{aligned}$$

$$\begin{aligned} \mathfrak{S}^{(3)}[\widetilde{I}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[-\frac{y^3}{9\varepsilon} + \left(-\frac{7y^3}{54} - \frac{4}{9}y^3\ln(2) + \frac{2y}{9} \right) + \varepsilon \left(\frac{575y^3}{81} + \frac{7\pi^2 y}{9} - 6y \right. \right. \\ & \left. \left. - \frac{26\pi^2 y^3}{27} - \frac{16}{3}y^3\ln^2(2) + \frac{145}{27}y^3\ln(2) + \frac{14}{3}y\ln^2(2) - \frac{40}{9}y\ln(2) \right) + O(\varepsilon^2) \right], \quad (9.21) \end{aligned}$$

$$\begin{aligned}
\mathcal{S}^{(4)}[\tilde{I}_{ij}] = & -\frac{\mathcal{N}_\varepsilon^2}{4\varepsilon E_{\max}^{4\varepsilon}} \left[\frac{1}{\varepsilon} \left(\frac{1}{12} - \frac{y^4}{12} \right) + \left(-\frac{13y^4}{72} - \frac{y^4}{3} \ln(2) + \frac{y^2}{6} - \frac{43}{360} + \frac{\ln(2)}{3} \right) \right. \\
& + \varepsilon \left(-2\pi^2 y^4 + \frac{2173y^4}{144} + \frac{13\pi^2 y^2}{6} - \frac{1183y^2}{72} - 15y^2 \ln(2) + \frac{27409}{10800} + \frac{349 \ln(2)}{180} \right. \\
& \left. \left. - \frac{5\pi^2}{18} - \frac{35}{3} y^4 \ln^2(2) + \frac{463}{36} y^4 \ln(2) + 13y^2 \ln^2(2) - 2 \ln^2(2) \right) + \mathcal{O}(\varepsilon^2) \right]. \quad (9.22)
\end{aligned}$$

We note that the dependence on $y = \cos \theta$ is as expected for a quantity that depends only on the scalar products in (7.26), which is a non-trivial consistency check besides the cancellation of spurious β -poles. For instance, terms linear in β in (9.14) and (9.19) must also be linear in y , and terms quadratic in β in (9.15) and (9.20) must either be independent of y or also quadratic in y . Most importantly, this expansion in β agrees with the result with exact β -dependence presented in [2] and, thus, provides a strong consistency check thereof.

CONCLUSION

In the absence of experimental signals of resonances pointing to new particles, the focus of the search for such particles is increasingly shifting to indirect probes. In view of the expected increasing experimental precision, it is therefore important to also improve the precision of theoretical predictions. Key challenges in this regard arise from QCD, which demands both calculations to high loop-orders in the perturbative regime and a careful separation of perturbative and non-perturbative dynamics. This thesis is split into two parts, each addressing issues in one of these categories.

In the first part, we analyzed the soft-overlap contribution to heavy-to-light transition form factors at large hadronic recoil. Specifically, we studied $B_c \rightarrow \eta_c$ transitions in a perturbative setup within the hierarchy $m_b \gg m_c \gg \Lambda_{\text{QCD}}$, which allowed us to understand the systematics of the double-logarithmic series to all orders. This setup serves as a template for cases involving non-perturbative dynamics. Instead of relying on EFT methods, which suffer from endpoint-divergent convolution integrals, we extracted double-logarithms diagrammatically by imposing *physical* cut-offs on loop momenta. We found that there are two distinct sources for double logarithms: on-shell soft-quark configurations obeying strong-rapidity ordering and on-shell soft-gluon configurations. Combining these two effects resums the double-logarithmic series in terms of a set of coupled integral equations given in (3.126) and (3.128). These integral equations represent the first successful resummation of double logarithms of the soft-overlap term. Additionally, they exhibit a novel set of features, combining exponential Sudakov factors with an iterative structure. We performed an independent check of the integral equations at fixed order up to the two-loop level based on a pole-cancellation argument derived from a bare factorization formula in SCET, which confirms the integral equations up to this accuracy. Despite the findings of this analysis it is important to develop a renormalizable EFT-based factorization theorem, since we consider our approach to be inapplicable if non-perturbative dynamics is involved, which is the case for phenomenologically relevant $B \rightarrow \pi$ or $B \rightarrow K$ transitions. The integral equations place important constraints on such a factorization theorem. Specifically, we expect that a factorization theorem must reflect the iterative nature of the integral equations. The development of such a factorization theorem and, in particular, its conceptual foundations are important for predictions of exclusive B decays that match the precision of future experimental measurements.

In the second part of this thesis, we calculated the phase-space integral of the double-emission eikonal for a massless and a massive emitter at an arbitrary angle to each other. This is a necessary ingredient to extend the nested soft-collinear subtraction scheme to cover massive final states at hadron colliders. We first performed the phase-space integral of the single-emission eikonal employing IBP identities and differential equation methods. The calculation of the double-emission eikonal is significantly more involved. We computed its expansion around the limit of small velocity of the massive emitter by expanding the differential equation of the MIs in this limit. This calculation constitutes an important check of the exact result with full dependence on the velocity presented in [2]. The equivalent calculation for two massive emitters in back-to-back kinematics with full dependence on the velocity was already performed in [56]. It would be an interesting direction for future research to investigate the phase-space integral of the double-emission eikonal with two massive emitters at an arbitrary angle to each other, which would extend the nested soft-collinear subtraction scheme e. g. to $t\bar{t}$ production in association with jets.

In summary, this thesis has addressed complementary challenges arising in the pursuit of high-precision predictions in QCD. The results obtained provide both concrete tools and conceptual guidance for future developments, thereby contributing to the broader program of improving theoretical predictions in line with experimental advances.

Part III

APPENDIX

In this Appendix we collect the MIs relevant for the leading ε -pole of the two-loop purely hard-collinear region of the form factor $F(\gamma)$ discussed in Sec. 4.3. In order to present them in a compact form we introduce abbreviations for the appearing denominators

$$D_1 = k_1^2, \quad D_2 = k_2^2, \quad (\text{A.1})$$

$$D_3 = k_{2-}, \quad D_4 = k_{1-}, \quad (\text{A.2})$$

$$D_5 = k_{2-} + k_{1-}, \quad D_6 = -p_{2-} + k_{1-}, \quad (\text{A.3})$$

$$D_7 = p_{2-} + k_{1-}, \quad D_8 = k_2^2 - k_{2-}\ell_+, \quad (\text{A.4})$$

$$D_9 = k_2^2 + k_{2-}\ell_+, \quad D_{10} = k_1^2 - \ell_+k_{1-}, \quad (\text{A.5})$$

$$D_{11} = k_1^2 + \ell_+k_{1-}, \quad D_{12} = k_{2-} - p_{\eta-}, \quad (\text{A.6})$$

$$D_{13} = -p_{\eta-} + k_{1-}, \quad D_{14} = k_2^2 - k_{2+}p_{2-}, \quad (\text{A.7})$$

$$D_{15} = k_2^2 + k_{2+}p_{2-}, \quad D_{16} = k_1^2 - p_{2-}k_{1+}, \quad (\text{A.8})$$

$$D_{17} = k_1^2 + p_{2-}k_{1+}, \quad D_{18} = k_2^2 - k_{2+}p_{1-}, \quad (\text{A.9})$$

$$D_{19} = k_2^2 + k_{2+}p_{1-}, \quad D_{20} = k_1^2 - k_{1+}p_{1-}, \quad (\text{A.10})$$

$$D_{21} = k_1^2 + k_{1+}p_{1-}, \quad D_{22} = -k_{2-} - p_{2-} + k_{1-}, \quad (\text{A.11})$$

$$D_{23} = k_{2-} + p_{2-} + k_{1-}, \quad D_{24} = k_{2-} + k_{1-} + p_{1-}, \quad (\text{A.12})$$

$$D_{25} = (k_1 - k_2)^2, \quad D_{26} = (k_1 + k_2)^2, \quad (\text{A.13})$$

$$D_{27} = k_1^2 - p_{\eta-}k_{1+}, \quad D_{28} = k_1^2 + p_{\eta-}k_{1+}, \quad (\text{A.14})$$

$$D_{29} = (k_1 - k_2)^2 - k_{2-}\ell_+ + \ell_+k_{1-}, \quad D_{30} = (k_1 + k_2)^2 + k_{2-}\ell_+ + \ell_+k_{1-}, \quad (\text{A.15})$$

$$D_{31} = (k_1 + k_2)^2 - k_{2-}\ell_+ - \ell_+k_{1-}, \quad D_{32} = (k_1 - k_2)^2 + k_{2-}\ell_+ - \ell_+k_{1-}, \quad (\text{A.16})$$

$$D_{33} = k_2^2 + k_{2-}\ell_+ - (k_{2+} + \ell_+)p_{2-}, \quad D_{34} = k_2^2 - k_{2-}\ell_+ + (k_{2+} - \ell_+)p_{2-}, \quad (\text{A.17})$$

$$D_{35} = k_2^2 - k_{2-}\ell_+ + (k_{2+} - \ell_+)p_{\eta-}, \quad D_{36} = k_1^2 + \ell_+(-p_{2-} + k_{1-}) - p_{2-}k_{1+}, \quad (\text{A.18})$$

$$D_{37} = k_1^2 + \ell_+(k_{1-} - p_{1-}) - k_{1+}p_{1-}, \quad D_{38} = k_2^2 + k_{2-}\ell_+ - (k_{2+} + \ell_+)p_{\eta-}, \quad (\text{A.19})$$

$$D_{39} = k_1^2 - \ell_+(p_{2-} + k_{1-}) + p_{2-}k_{1+}, \quad D_{40} = (k_1 - k_2)^2 - k_{2+}p_{2-} + p_{2-}k_{1+}, \quad (\text{A.20})$$

$$D_{41} = (k_1 + k_2)^2 + k_{2+}p_{2-} + p_{2-}k_{1+}, \quad D_{42} = (k_1 - k_2)^2 + k_{2+}p_{1-} - k_{1+}p_{1-}, \quad (\text{A.21})$$

$$D_{43} = (k_1 + k_2)^2 + k_{2+}p_{1-} + k_{1+}p_{1-}, \quad D_{44} = (k_1 - k_2)^2 + k_{2+}p_{2-} - p_{2-}k_{1+}, \quad (\text{A.22})$$

$$D_{45} = (k_1 + k_2)^2 - k_{2+}p_{1-} - k_{1+}p_{1-}, \quad D_{46} = (k_1 - k_2)^2 - k_{2+}p_{1-} + k_{1+}p_{1-}, \quad (\text{A.23})$$

$$D_{47} = k_1^2 + \ell_+(-p_{\eta-} + k_{1-}) - p_{\eta-}k_{1+}, \quad D_{48} = k_1^2 - \ell_+(p_{\eta-} + k_{1-}) + p_{\eta-}k_{1+}, \quad (\text{A.24})$$

$$D_{49} = (k_1 - k_2)^2 - k_{2+}p_{\eta-} + p_{\eta-}k_{1+}, \quad D_{50} = (k_1 + k_2)^2 + k_{2+}p_{\eta-} + p_{\eta-}k_{1+}, \quad (\text{A.25})$$

$$D_{51} = (k_1 + k_2)^2 - k_{2+}p_{\eta-} - p_{\eta-}k_{1+}, \quad (\text{A.26})$$

$$D_{52} = (k_1 + k_2)^2 + k_{2-}\ell_+ - k_{2+}p_{2-} - \ell_+p_{2-} + \ell_+k_{1-} - p_{2-}k_{1+}, \quad (\text{A.27})$$

$$D_{53} = (k_1 - k_2)^2 - k_{2-}\ell_+ + k_{2+}p_{2-} - \ell_+p_{2-} + \ell_+k_{1-} - p_{2-}k_{1+}, \quad (\text{A.28})$$

$$D_{54} = (k_1 - k_2)^2 + k_{2-}\ell_+ - k_{2+}p_{2-} - \ell_+p_{2-} - \ell_+k_{1-} + p_{2-}k_{1+}, \quad (\text{A.29})$$

$$D_{55} = (k_1 + k_2)^2 - k_{2-}\ell_+ + k_{2+}p_{2-} - \ell_+p_{2-} - \ell_+k_{1-} + p_{2-}k_{1+}, \quad (\text{A.30})$$

$$D_{56} = (k_1 + k_2)^2 + k_{2-}\ell_+ + \ell_+(k_{1-} - p_{1-}) - k_{2+}p_{1-} - k_{1+}p_{1-}, \quad (\text{A.31})$$

$$D_{57} = (k_1 - k_2)^2 + k_{2-}\ell_+ - \ell_+(k_{1-} + p_{1-}) - k_{2+}p_{1-} + k_{1+}p_{1-}, \quad (\text{A.32})$$

$$D_{58} = (k_1 + k_2)^2 - k_{2-}\ell_+ - \ell_+(k_{1-} + p_{1-}) + k_{2+}p_{1-} + k_{1+}p_{1-}, \quad (\text{A.33})$$

$$D_{59} = (k_1 + k_2)^2 + k_{2-}\ell_+ - k_{2+}p_{\eta-} - \ell_+p_{\eta-} + \ell_+k_{1-} - p_{\eta-}k_{1+}, \quad (\text{A.34})$$

$$D_{60} = (k_1 - k_2)^2 - k_{2-}\ell_+ + k_{2+}p_{\eta-} - \ell_+p_{\eta-} + \ell_+k_{1-} - p_{\eta-}k_{1+}, \quad (\text{A.35})$$

$$D_{61} = (k_1 - k_2)^2 + k_{2-}\ell_+ - k_{2+}p_{\eta-} - \ell_+p_{\eta-} - \ell_+k_{1-} + p_{\eta-}k_{1+}, \quad (\text{A.36})$$

$$D_{62} = (k_1 + k_2)^2 - k_{2-}\ell_+ + k_{2+}p_{\eta-} - \ell_+p_{\eta-} - \ell_+k_{1-} + p_{\eta-}k_{1+}. \quad (\text{A.37})$$

Given these denominators the MIs can be written as

$$\tilde{B}_1 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_9 D_{15} D_{21} D_{48} D_{57}}, \quad \tilde{B}_2 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_3 D_{11} D_{15} D_{21} D_{53}}, \quad (\text{A.38})$$

$$\tilde{B}_3 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_4 D_{15} D_{20} D_{23} D_{56}}, \quad \tilde{B}_4 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_4 D_9 D_{15} D_{20} D_{56}}, \quad (\text{A.39})$$

$$\tilde{B}_5 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_{10} D_{15} D_{20} D_{35} D_{55}}, \quad \tilde{B}_6 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_3 D_{10} D_{20} D_{35} D_{42}}, \quad (\text{A.40})$$

$$\tilde{B}_7 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_4 D_{12} D_{20} D_{33} D_{59}}, \quad \tilde{B}_8 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_4 D_9 D_{15} D_{36} D_{44}}, \quad (\text{A.41})$$

$$\tilde{B}_9 = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_4 D_{19} D_{33} D_{43} D_{48}}, \quad \tilde{B}_{10} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_3 D_{13} D_{36} D_{53} D_{60}}, \quad (\text{A.42})$$

$$\tilde{B}_{11} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_3 D_{10} D_{38} D_{39} D_{54}}, \quad \tilde{B}_{12} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_3 D_4 D_{17} D_{26} D_{38} D_{45}}, \quad (\text{A.43})$$

$$\tilde{B}_{13} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_5 D_8^2 D_{14} D_{51}}, \quad \tilde{B}_{14} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_8 D_{18} D_{31} D_{48} D_{55}}, \quad (\text{A.44})$$

$$\tilde{B}_{15} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_3 D_{28} D_{38} D_{49}}, \quad \tilde{B}_{16} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_3 D_{11} D_{28} D_{38} D_{49}}, \quad (\text{A.45})$$

$$\tilde{B}_{17} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_3 D_{11} D_{17} D_{38} D_{49}}, \quad \tilde{B}_{18} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{16} D_{19} D_{24} D_{58}}, \quad (\text{A.46})$$

$$\tilde{B}_{19} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_9 D_{17} D_{19} D_{48}}, \quad \tilde{B}_{20} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_9 D_{17} D_{19} D_{48}}, \quad (\text{A.47})$$

$$\tilde{B}_{21} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_9 D_{17} D_{19} D_{48} D_{54}}, \quad \tilde{B}_{22} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_8 D_{17} D_{48} D_{55} D_{62}}, \quad (\text{A.48})$$

$$\tilde{B}_{23} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_8 D_{17} D_{48} D_{62}}, \quad \tilde{B}_{24} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_8 D_{17} D_{48} D_{55} D_{62}}, \quad (\text{A.49})$$

$$\tilde{B}_{25} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{21} D_{48} D_{49}}, \quad \tilde{B}_{26} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_8 D_{14} D_{21} D_{48} D_{49}}, \quad (\text{A.50})$$

$$\tilde{B}_{27} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{15} D_{21} D_{57}}, \quad \tilde{B}_{28} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{21} D_{48} D_{57}}, \quad (\text{A.51})$$

$$\tilde{B}_{29} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_9 D_{15} D_{21} D_{48} D_{57}}, \quad \tilde{B}_{30} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_3 D_{11} D_{15} D_{21} D_{53}}, \quad (\text{A.52})$$

$$\tilde{B}_{31} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_3 D_6 D_{11} D_{15} D_{21} D_{53}}, \quad \tilde{B}_{32} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_3 D_{11} D_{21} D_{30} D_{52}}, \quad (\text{A.53})$$

$$\tilde{B}_{33} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_3 D_6 D_{11} D_{21} D_{30} D_{52}}, \quad \tilde{B}_{34} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_3 D_{10} D_{20} D_{35}^2 D_{42}}, \quad (\text{A.54})$$

$$\tilde{B}_{35} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_3 D_{10} D_{20} D_{38} D_{39} D_{54}}, \quad \tilde{B}_{36} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{41} D_{48} D_{50}}, \quad (\text{A.55})$$

$$\tilde{B}_{37} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{15} D_{48} D_{50}}, \quad \tilde{B}_{38} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_9 D_{15} D_{41} D_{48} D_{50}}, \quad (\text{A.56})$$

$$\tilde{B}_{39} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_8 D_{14} D_{31} D_{48} D_{58}}, \quad \tilde{B}_{40} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_9 D_{15} D_{36} D_{44} D_{47}}, \quad (\text{A.57})$$

$$\tilde{B}_{41} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_8 D_{18} D_{31} D_{48}}, \quad \tilde{B}_{42} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_8 D_{18} D_{31} D_{48} D_{55}}, \quad (\text{A.58})$$

$$\tilde{B}_{43} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_8 D_{18} D_{31}^2 D_{48}}, \quad \tilde{B}_{44} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_{21} D_{26} D_{38} D_{52}}, \quad (\text{A.59})$$

$$\tilde{B}_{45} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_{10} D_{15} D_{26} D_{55}}, \quad \tilde{B}_{46} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_8 D_{16} D_{46} D_{53}}, \quad (\text{A.60})$$

$$\tilde{B}_{47} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{11} D_{17} D_{30} D_{38}}, \quad \tilde{B}_{48} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_8 D_{46} D_{53}}, \quad (\text{A.61})$$

$$\tilde{B}_{49} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_8 D_{21} D_{46} D_{53}}, \quad \tilde{B}_{50} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{19} D_{26} D_{47}}, \quad (\text{A.62})$$

$$\tilde{B}_{51} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{26} D_{34}^2 D_{47}}, \quad \tilde{B}_{52} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{16} D_{19} D_{60}}, \quad (\text{A.63})$$

$$\tilde{B}_{53} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{12} D_{16} D_{57}^2}, \quad \tilde{B}_{54} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{16} D_{22} D_{32}}, \quad (\text{A.64})$$

$$\tilde{B}_{55} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_5^2 D_{11} D_{17} D_{50}}, \quad \tilde{B}_{56} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_3 D_{11} D_{17} D_{50}}, \quad (\text{A.65})$$

$$\tilde{B}_{57} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_4 D_8 D_{17} D_{40} D_{48}}, \quad \tilde{B}_{58} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{21} D_{34} D_{62}}, \quad (\text{A.66})$$

$$\tilde{B}_{59} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{21} D_{34} D_{48}},$$

$$\tilde{B}_{61} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{20} D_{38}},$$

$$\tilde{B}_{63} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{20} D_{23} D_{45}},$$

$$\tilde{B}_{65} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_7 D_{10} D_{20} D_{54}},$$

$$\tilde{B}_{67} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_3 D_{21} D_{26} D_{38} D_{48}},$$

$$\tilde{B}_{69} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_3 D_{10} D_{16} D_{32} D_{38}},$$

$$\tilde{B}_{71} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_{33} D_{39}},$$

$$\tilde{B}_{73} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{16} D_{38}},$$

$$\tilde{B}_{75} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_{36} D_{44}},$$

$$\tilde{B}_{77} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{16} D_{58}},$$

$$\tilde{B}_{79} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{17} D_{48} D_{54}},$$

$$\tilde{B}_{81} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{21} D_{29} D_{36}},$$

$$\tilde{B}_{83} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{20} D_{59}},$$

$$\tilde{B}_{85} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{20} D_{55}},$$

$$\tilde{B}_{87} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{44} D_{47}},$$

$$\tilde{B}_{60} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{13} D_{15} D_{20} D_{60}}, \quad (\text{A.67})$$

$$\tilde{B}_{62} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{20} D_{38} D_{45}}, \quad (\text{A.68})$$

$$\tilde{B}_{64} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_3 D_{10} D_{20} D_{54}}, \quad (\text{A.69})$$

$$\tilde{B}_{66} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{23} D_{39} D_{45}}, \quad (\text{A.70})$$

$$\tilde{B}_{68} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_3 D_{10} D_{27} D_{32} D_{38}}, \quad (\text{A.71})$$

$$\tilde{B}_{70} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_3 D_{10} D_{16} D_{32} D_{38}^2}, \quad (\text{A.72})$$

$$\tilde{B}_{72} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_1 D_2 D_{35} D_{47}}, \quad (\text{A.73})$$

$$\tilde{B}_{74} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{21} D_{36} D_{44}}, \quad (\text{A.74})$$

$$\tilde{B}_{76} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{20} D_{25} D_{47}}, \quad (\text{A.75})$$

$$\tilde{B}_{78} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{17} D_{61}}, \quad (\text{A.76})$$

$$\tilde{B}_{80} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{17} D_{29} D_{37}}, \quad (\text{A.77})$$

$$\tilde{B}_{82} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{43} D_{48}}, \quad (\text{A.78})$$

$$\tilde{B}_{84} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{20} D_{59}}, \quad (\text{A.79})$$

$$\tilde{B}_{86} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_{10} D_{20} D_{55}}, \quad (\text{A.80})$$

$$\tilde{B}_{88} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{1}{D_2 D_4 D_{44} D_{47}}. \quad (\text{A.81})$$

In this Appendix we derive a one-fold integral equation for $g_{2b}(\rho, \rho)$ similar to (5.27). In analogy to (5.24), we have for $g_{2b}(\rho, \eta)$

$$g_{2b}(\rho, \eta) = \left(\int_0^\eta d\eta' \eta' + \int_\eta^\rho d\eta' \eta \right) e^{-\eta'(\rho-\eta')} C_{2b}(\rho, \eta'). \quad (\text{B.1})$$

Equating the two expression for $g_{2b}(\rho, \eta)$ in (5.46) and (B.1), we obtain

$$g_{2b}(\rho, \rho) = \int_0^\rho d\eta' \eta' e^{-\eta'(\rho-\eta')} C_{2b}(\rho, \eta'). \quad (\text{B.2})$$

The form of (B.2) is compact, but $C_{2b}(\rho, \eta')$ is only determined up to $C_{2b}(\eta', \eta')$. Plugging (5.45) into (B.2) yields

$$\begin{aligned} g_{2b}(\rho, \rho) = & \int_0^\rho d\eta' \eta' e^{-\eta'(\rho-\eta')} (g_{2b}(\eta', \eta') + 1) - \int_0^\rho d\eta' \eta' e^{-\eta'(\rho-\eta')} C_1(\eta') \\ & + \int_0^\rho d\eta' \int_{\eta'}^\rho d\eta'' \int_{\eta''}^\rho d\rho' \eta' e^{-\eta'(\rho-\rho')} e^{-\eta''(\rho'-\eta'')} C_1(\eta''). \end{aligned} \quad (\text{B.3})$$

The last line in (B.3)

$$\tilde{g}_{2b}(\rho, \rho) \equiv \int_0^\rho d\eta' \int_{\eta'}^\rho d\eta'' \int_{\eta''}^\rho d\rho' \eta' e^{-\eta'(\rho-\rho')} e^{-\eta''(\rho'-\eta'')} C_1(\eta''), \quad (\text{B.4})$$

requires some more work. Changing the order of integration for the η' and η'' integrals

$$\int_0^\rho d\eta' \int_{\eta'}^\rho d\eta'' = \int_0^\rho d\eta'' \int_0^{\eta''} d\eta', \quad (\text{B.5})$$

and evaluating the ρ' -integral

$$\int_{\eta''}^\rho d\rho' e^{-\eta'(\rho-\rho')} e^{-\eta''(\rho'-\eta'')} = \frac{e^{-\eta''(\rho-\eta'')} - e^{-\eta'(\rho-\eta'')}}{\eta' - \eta''}, \quad (\text{B.6})$$

gives

$$\tilde{g}_{2b}(\rho, \rho) = \int_0^\rho d\eta'' \int_0^{\eta''} d\eta' \frac{\eta' e^{-\eta''(\rho-\eta'')} - \eta' e^{-\eta'(\rho-\eta'')}}{\eta' - \eta''} C_1(\eta''). \quad (\text{B.7})$$

Renaming η'' to ρ' and multiplying numerator and denominator with -1 leads to

$$\tilde{g}_{2b}(\rho, \rho) = \int_0^\rho d\rho' \int_0^{\rho'} d\eta' \frac{\eta' e^{-\eta'(\rho-\rho')} - \eta' e^{-\rho'(\rho-\rho')}}{\rho' - \eta'} C_1(\rho'). \quad (\text{B.8})$$

In the second term of the numerator in (B.8) we replace η' with ρ' . Then, we have to account for the difference

$$\int_0^\rho d\rho' \int_0^{\rho'} d\eta' \frac{(\rho' - \eta') e^{-\rho'(\rho-\rho')}}{\rho' - \eta'} C_1(\rho') = \int_0^\rho d\rho' \rho' e^{-\rho'(\rho-\rho')} C_1(\rho'), \quad (\text{B.9})$$

which cancels the second term in the first line of (B.3). Summarizing the above steps we obtain

$$g_{2b}(\rho, \rho) = \int_0^\rho d\eta' \, \eta' e^{-\eta'(\rho-\eta')} (g_{2b}(\eta', \eta') + 1) + \int_0^\rho d\rho' \int_0^{\rho'} d\eta' \frac{\eta' e^{-\eta'(\rho-\rho')} - \rho' e^{-\rho'(\rho-\rho')}}{\rho' - \eta'} C_1(\rho'). \quad (\text{B.10})$$

In the first line of (B.10) we replace $\eta' = \rho'$ and in the second line we change variables $\eta' = x' / (\rho - \rho')$ to get

$$g_{2b}(\rho, \rho) = \int_0^\rho d\rho' \, \rho' e^{-\rho'(\rho-\rho')} (g_{2b}(\rho', \rho') + 1) + \int_0^\rho d\rho' \, \rho' \int_0^{\rho'(\rho-\rho')} dx' \frac{x' e^{-x'} - \rho'(\rho-\rho') e^{-\rho'(\rho-\rho')}}{\rho'(\rho-\rho')(\rho'(\rho-\rho') - x')} C_1(\rho'). \quad (\text{B.11})$$

Finally, the integration kernel in the last line of (B.11) can be written in a compact form

$$1 + g_{2b}(\rho, \rho) = 1 + \int_0^\rho d\rho' \, \rho' e^{-\rho'(\rho-\rho')} \left(1 + g_{2b}(\rho', \rho') \right) + \int_0^\rho d\rho' \, \rho' \mathcal{K}(\rho'(\rho-\rho')) C_1(\rho'), \quad (\text{B.12})$$

where the integration kernel $\mathcal{K}(x)$ is defined as

$$\mathcal{K}(x) = \int_0^\infty dx' \left[\frac{\theta(x-x')}{x(x-x')} \right]_+ x' e^{-x'}, \quad (\text{B.13})$$

and the plus distribution (cf. (1.111)) subtracts the limit $x' \rightarrow x$.

COLOR NOTATION

We use the color space formalism introduced by Catani [153] in the context of QCD amplitudes and IR subtraction at higher loop orders. This formalism treats scattering amplitudes as vectors in color space, and uses color charge operators to encode color correlations between external partons. It has the advantage that the details of color correlations can be left implicit. For instance, the leading soft limit is universal with respect to the emitting parton up to color which can be denoted by the color operators defined below (cf. e. g. (7.4), (7.5) and (7.6)). For example, for some basis $\langle c_1, \dots, c_n |$ in color space we write

$$M_{c_1, \dots, c_n} = \langle c_1, \dots, c_n | M \rangle . \quad (\text{C.1})$$

We define color operators $\mathbf{T}_i$ such that

$$\mathbf{T}_i \equiv T_i^a |a\rangle . \quad (\text{C.2})$$

The color generator acts in color space as

$$\langle a, c_1, \dots, c_n | \mathbf{T}_i | b_1, \dots, b_n \rangle = \delta_{c_1 b_1} \dots T_{c_i b_i}^a \dots \delta_{c_n b_n} , \quad (\text{C.3})$$

where

$$T_{cb}^a \equiv \begin{cases} if_{cab} , & \text{if parton } i \text{ is a gluon ,} \\ t_{cb}^a , & \text{if parton } i \text{ is an outgoing quark or incoming antiquark ,} \\ \bar{t}_{cb}^a = -t_{bc}^a , & \text{if parton } i \text{ is an outgoing antiquark or incoming quark ,} \end{cases} \quad (\text{C.4})$$

f_{cab} are the $SU(N)$ structure constants and t^a are the $SU(N)$ generators in the fundamental representation. These operators obey the standard color algebra

$$[\mathbf{T}_i^a, \mathbf{T}_j^b] = if^{abc} \delta_{ij} \mathbf{T}_i^c , \quad (\text{C.5})$$

and they satisfy overall color conservation

$$\sum_i \mathbf{T}_i = 0 . \quad (\text{C.6})$$

A important object in this formalism that appears, for instance, in (7.6) is the color dipole operator

$$\mathbf{T}_i \cdot \mathbf{T}_j = \sum_a \mathbf{T}_i^a \mathbf{T}_j^a , \quad (\text{C.7})$$

which encodes the color correlation between external legs i and j .

MASTER INTEGRALS, ANGULAR INTEGRALS AND BOUNDARY CONDITIONS

D.1 MASTER INTEGRALS FOR $\mathfrak{S} [\Xi_{ij}]$

The master integrals for $\mathfrak{S} [\Xi_{ij}]$ are

$$J_1 = \Delta_{0,0,1}^{0,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_n \cdot p_j)}, \quad (D.1)$$

$$J_2 = \Delta_{0,0,2}^{0,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^2(l_n \cdot p_j)}, \quad (D.2)$$

$$J_3 = \Delta_{1,0,1}^{0,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_i)(l_n \cdot p_j)}, \quad (D.3)$$

$$J_4 = \Delta_{0,0,1}^{1,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_m \cdot p_j)(l_n \cdot p_j)}, \quad (D.4)$$

$$J_5 = \Delta_{1,0,1}^{1,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_i)(l_m \cdot p_j)(l_n \cdot p_j)}, \quad (D.5)$$

$$J_6 = \Delta_{0,0,1}^{0,1,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_n \cdot p_j)(l_m \cdot l_n)}, \quad (D.6)$$

$$J_7 = \Delta_{0,0,2}^{0,1,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^2(l_n \cdot p_j)(l_m \cdot l_n)}, \quad (D.7)$$

$$J_8 = \Delta_{1,0,0}^{0,1,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_j)(l_m \cdot l_n)}, \quad (D.8)$$

$$J_9 = \Delta_{2,0,0}^{0,1,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)^2(l_n \cdot p_j)(l_m \cdot l_n)}, \quad (D.9)$$

$$J_{10} = \Delta_{0,0,1}^{0,2,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_n \cdot p_j)^2(l_m \cdot l_n)}, \quad (D.10)$$

$$J_{11} = \Delta_{1,0,0}^{0,2,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_j)^2(l_m \cdot l_n)}, \quad (D.11)$$

$$J_{12} = \Delta_{0,1,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i)(l_{mn} \cdot p_j)}, \quad (D.12)$$

$$J_{13} = \Delta_{0,2,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i)^2(l_{mn} \cdot p_j)}, \quad (D.13)$$

$$J_{14} = \Delta_{1,1,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_i)(l_{mn} \cdot p_j)}, \quad (D.14)$$

$$J_{15} = \Delta_{1,2,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_i)^2(l_{mn} \cdot p_j)}, \quad (D.15)$$

$$J_{16} = \Delta_{2,1,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)^2 (l_n \cdot p_i) (l_{mn} \cdot p_j)}, \quad (D.16)$$

$$J_{17} = \Delta_{0,1,0}^{0,0,2}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i) (l_{mn} \cdot p_j)^2}, \quad (D.17)$$

$$J_{18} = \Delta_{1,1,0}^{1,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i) (l_n \cdot p_i) (l_m \cdot p_j) (l_{mn} \cdot p_j)}, \quad (D.18)$$

$$J_{19} = \Delta_{0,1,0}^{0,0,1}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i) (l_{mn} \cdot p_j) (l_m \cdot l_n)}, \quad (D.19)$$

$$J_{20} = \Delta_{0,2,0}^{0,0,1}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i)^2 (l_{mn} \cdot p_j) (l_m \cdot l_n)}, \quad (D.20)$$

$$J_{21} = \Delta_{0,1,0}^{0,0,2}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i) (l_{mn} \cdot p_j)^2 (l_m \cdot l_n)}, \quad (D.21)$$

$$J_{22} = \Delta_{0,1,0}^{1,0,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i) (l_m \cdot p_j) (l_m \cdot l_n)}, \quad (D.22)$$

$$J_{23} = \Delta_{0,2,0}^{1,0,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i)^2 (l_m \cdot p_j) (l_m \cdot l_n)}, \quad (D.23)$$

$$J_{24} = \Delta_{0,1,0}^{2,0,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i) (l_m \cdot p_j)^2 (l_m \cdot l_n)}, \quad (D.24)$$

$$J_{25} = \Delta_{0,0,1}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)}, \quad (D.25)$$

$$J_{26} = \Delta_{0,0,2}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^2}, \quad (D.26)$$

$$J_{27} = \Delta_{0,0,3}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^3}, \quad (D.27)$$

$$J_{28} = \Delta_{1,0,1}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i) (l_{mn} \cdot p_i)}, \quad (D.28)$$

$$J_{29} = \Delta_{0,0,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_j)}, \quad (D.29)$$

$$J_{30} = \Delta_{0,0,1}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i) (l_{mn} \cdot p_j)}, \quad (D.30)$$

$$J_{31} = \Delta_{0,0,2}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^2 (l_{mn} \cdot p_j)}, \quad (D.31)$$

$$J_{32} = \Delta_{0,0,3}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^3 (l_{mn} \cdot p_j)}, \quad (D.32)$$

$$J_{33} = \Delta_{1,0,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i) (l_{mn} \cdot p_j)}, \quad (D.33)$$

$$J_{34} = \Delta_{1,0,1}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i) (l_{mn} \cdot p_i) (l_{mn} \cdot p_j)}, \quad (D.34)$$

$$J_{35} = \Delta_{1,0,2}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_i)^2(l_{mn} \cdot p_j)}, \quad (D.35)$$

$$J_{36} = \Delta_{2,0,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)^2(l_{mn} \cdot p_j)}, \quad (D.36)$$

$$J_{37} = \Delta_{2,0,1}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)^2(l_{mn} \cdot p_i)(l_{mn} \cdot p_j)}, \quad (D.37)$$

$$J_{38} = \Delta_{0,0,0}^{0,0,2}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_j)^2}, \quad (D.38)$$

$$J_{39} = \Delta_{1,0,0}^{0,0,2}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_j)^2}, \quad (D.39)$$

$$J_{40} = \Delta_{0,0,1}^{1,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_m \cdot p_j)}, \quad (D.40)$$

$$J_{41} = \Delta_{0,0,2}^{1,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^2(l_m \cdot p_j)}, \quad (D.41)$$

$$J_{42} = \Delta_{1,0,1}^{1,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_i)(l_m \cdot p_j)}, \quad (D.42)$$

$$J_{43} = \Delta_{0,0,1}^{1,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_m \cdot p_j)(l_{mn} \cdot p_j)}, \quad (D.43)$$

$$J_{44} = \Delta_{1,0,1}^{1,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_i)(l_m \cdot p_j)(l_{mn} \cdot p_j)}, \quad (D.44)$$

$$J_{45} = \Delta_{1,0,0}^{0,0,1}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_j)(l_m \cdot l_n)}, \quad (D.45)$$

$$J_{46} = \Delta_{2,0,0}^{0,0,1}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)^2(l_{mn} \cdot p_j)(l_m \cdot l_n)}, \quad (D.46)$$

$$J_{47} = \Delta_{1,0,0}^{0,0,2}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)(l_{mn} \cdot p_j)^2(l_m \cdot l_n)}, \quad (D.47)$$

$$J_{48} = \Delta_{0,0,1}^{1,0,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_m \cdot p_j)(l_m \cdot l_n)}, \quad (D.48)$$

$$J_{49} = \Delta_{0,0,2}^{1,0,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)^2(l_m \cdot p_j)(l_m \cdot l_n)}, \quad (D.49)$$

$$J_{50} = \Delta_{0,0,1}^{2,0,0}(1) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_i)(l_m \cdot p_j)^2(l_m \cdot l_n)}, \quad (D.50)$$

$$J_{51} = \Delta_{0,0,0}^{0,0,0}(0) = \int [dl_m][dl_n] \delta(1-l_m \cdot P) \delta(z-l_n \cdot P), \quad (D.51)$$

$$J_{52} = \Delta_{0,1,0}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_n \cdot p_i)}, \quad (D.52)$$

$$J_{53} = \Delta_{1,0,0}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_m \cdot p_i)}, \quad (D.53)$$

$$J_{54} = \Delta_{1,1,0}^{0,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_i)}, \quad (D.54)$$

$$J_{55} = \Delta_{0,1,0}^{0,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_n \cdot p_i)(l_n \cdot p_j)}, \quad (D.55)$$

$$J_{56} = \Delta_{1,1,0}^{0,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_i)(l_n \cdot p_j)}, \quad (D.56)$$

$$J_{57} = \Delta_{1,0,0}^{1,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_m \cdot p_i)(l_m \cdot p_j)}, \quad (D.57)$$

$$J_{58} = \Delta_{1,1,0}^{1,0,0}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_i)(l_m \cdot p_j)}, \quad (D.58)$$

$$J_{59} = \Delta_{1,1,0}^{1,1,0}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_m \cdot p_i)(l_n \cdot p_i)(l_m \cdot p_j)(l_n \cdot p_j)}. \quad (D.59)$$

D.2 MASTER INTEGRALS FOR THE BOUNDARY AT $\beta = 0$

The MIs at $\beta = 0$ are J_{51} , J_{29} , J_{38} . For the first integral we can simply write

$$J_{51} = \Delta_{0,0,0}^{0,0,0}(0) = \int [dl_m][dl_n] \delta(1 - l_m \cdot P) \delta(z - l_n \cdot P) = \mathcal{N}_\varepsilon^2 z^{1-2\varepsilon}. \quad (D.60)$$

For the second integral we have to introduce angles between the loop momenta and p_j . A single angle for a given integral over a $(d-1)$ -dimensional sphere can be parameterized as

$$\int d\Omega_m^{(d-1)} f(\vec{n}_m \cdot \vec{n}_j) = \Omega^{(d-2)} \int_{-1}^1 d\cos\theta_1 (\sin\theta_1)^{-2\varepsilon} f(\cos\theta_1), \quad (D.61)$$

where $\Omega^{(d)} = 2\pi^{d/2}/\Gamma(d/2)$ is the d -dimensional solid angle and f is an arbitrary function. Implementing (D.61) for both required angles, we obtain

$$\begin{aligned} J_{29} &= \Delta_{0,0,0}^{0,0,1}(0) = \int [dl_m][dl_n] \frac{\delta(1 - l_m \cdot P) \delta(z - l_n \cdot P)}{(l_{mn} \cdot p_j)} \\ &= \frac{\mathcal{N}_\varepsilon^2 z^{1-2\varepsilon} \Gamma\left(\frac{3}{2} - \varepsilon\right)^2}{\pi \Gamma(1 - \varepsilon)^2} \int_{-1}^1 d\cos\theta_1 \int_{-1}^1 d\cos\theta_2 \frac{\sin^{-2\varepsilon}\theta_1 \sin^{-2\varepsilon}\theta_2}{1 - \cos\theta_1 + z(1 - \cos\theta_2)}, \end{aligned} \quad (D.62)$$

where $\cos\theta_1 = \vec{n}_m \cdot \vec{n}_j$ and $\cos\theta_2 = \vec{n}_n \cdot \vec{n}_j$. Changing the variables according to $\cos\theta_k = 1 - 2\xi_k$ for both angles, we arrive at

$$J_{29} = \frac{\mathcal{N}_\varepsilon^2 z^{1-2\varepsilon} \Gamma\left(\frac{3}{2} - \varepsilon\right)^2}{\pi \Gamma(1 - \varepsilon)^2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \frac{2^{1-4\varepsilon} ((1 - \xi_1)\xi_1)^{-\varepsilon} ((1 - \xi_2)\xi_2)^{-\varepsilon}}{\xi_1 + \xi_2 z}. \quad (D.63)$$

This integral can be solved by direct computation and the result is

$$\begin{aligned} J_{29} &= \frac{1}{2} \mathcal{N}_\varepsilon^2 z^{-2\varepsilon} \left(\frac{(2\varepsilon - 1) {}_3F_2\left(1, 1 - \varepsilon, 2\varepsilon; 2 - 2\varepsilon, \varepsilon + 1; -\frac{1}{z}\right)}{\varepsilon} \right. \\ &\quad \left. + \frac{2^{1-4\varepsilon} z^\varepsilon \Gamma(2 - 2\varepsilon) \Gamma\left(\frac{1}{2} - \varepsilon\right) \Gamma\left(\frac{3}{2} - \varepsilon\right) \Gamma(\varepsilon) {}_2F_1\left(1 - 2\varepsilon, \varepsilon; 2 - 3\varepsilon; -\frac{1}{z}\right)}{\pi \Gamma(2 - 3\varepsilon)} \right). \end{aligned} \quad (D.64)$$

When integrating over z , the limit $z \rightarrow 0$ is divergent. Therefore, we have to integrate the leading terms in this limit with full dependence on ε . It is convenient to extract the leading terms by means of the identities

$${}_2F_1\left(1-2\varepsilon, \varepsilon; 2-3\varepsilon; -\frac{1}{z}\right) = \frac{z^{1-2\varepsilon}\Gamma(3\varepsilon-1)\Gamma(2-3\varepsilon) {}_2F_1(1-2\varepsilon, \varepsilon; 2-3\varepsilon; -z)}{\Gamma(1-\varepsilon)\Gamma(\varepsilon)} + \frac{z^\varepsilon\Gamma(1-3\varepsilon)\Gamma(2-3\varepsilon) {}_2F_1(\varepsilon, 4\varepsilon-1; 3\varepsilon; -z)}{\Gamma(2-4\varepsilon)\Gamma(1-2\varepsilon)}, \quad (\text{D.65})$$

and

$${}_3F_2\left(1, 1-\varepsilon, 2\varepsilon; 2-2\varepsilon, \varepsilon+1; -\frac{1}{z}\right) = \frac{\Gamma(2-2\varepsilon)\Gamma(\varepsilon+1)}{\Gamma(1-\varepsilon)\Gamma(2\varepsilon)} \left(\frac{z^{1-\varepsilon}\Gamma(\varepsilon)\Gamma(3\varepsilon-1) {}_2F_1(1-2\varepsilon, \varepsilon; 2-3\varepsilon; -z)}{\Gamma(2\varepsilon)} + \frac{z^{2\varepsilon}\Gamma(2\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1-3\varepsilon) {}_2F_1(\varepsilon, 4\varepsilon-1; 3\varepsilon; -z)}{\Gamma(2-4\varepsilon)\Gamma(1-\varepsilon)} + \frac{z\Gamma(-\varepsilon)\Gamma(2\varepsilon-1) {}_3F_2(1, 1-\varepsilon, 2\varepsilon; 2-2\varepsilon, \varepsilon+1; -z)}{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)} \right). \quad (\text{D.66})$$

For

$$J_{38} = \Delta_{0,0,0}^{0,0,2}(0) = \int [dl_m][dl_n] \frac{\delta(1-l_m \cdot P) \delta(z-l_n \cdot P)}{(l_{mn} \cdot p_j)^2}, \quad (\text{D.67})$$

we repeat the steps leading to (D.63) and obtain

$$J_{38} = \frac{\mathcal{N}_\varepsilon^2 z^{1-2\varepsilon} \Gamma\left(\frac{3}{2}-\varepsilon\right)^2}{\pi \Gamma(1-\varepsilon)^2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \frac{2^{-4\varepsilon} ((1-\xi_1)\xi_1)^{-\varepsilon} ((1-\xi_2)\xi_2)^{-\varepsilon}}{(\xi_1 + \xi_2 z)^2}, \quad (\text{D.68})$$

where the only difference is the power of the denominator. Here, we have to use the Mellin-Barnes technique to factorize the integrals over ξ_1 and ξ_2

$$\frac{1}{(A_1 + A_2)^\lambda} = \int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i} \frac{\Gamma(-x)\Gamma(\lambda+x)}{\Gamma(\lambda)} A_1^x A_2^{-\lambda-x}, \quad (\text{D.69})$$

where c has to be chosen such that the integration contour separates the series of poles of the Γ functions going to plus and minus infinity. Finally, the integral over x yields

$$J_{38} = \frac{1}{2} \mathcal{N}_\varepsilon^2 z^{-3\varepsilon} \left(\frac{(2\varepsilon-1)z^{\varepsilon+1} {}_3F_2(2, 1-\varepsilon, 2\varepsilon+1; 2-2\varepsilon, \varepsilon+2; -z)}{\varepsilon+1} - \frac{2^{-4\varepsilon}\Gamma(1-2\varepsilon)\Gamma\left(\frac{3}{2}-\varepsilon\right)^2\Gamma(\varepsilon)}{\pi\Gamma(1-3\varepsilon)} {}_3F_2(1-\varepsilon, -2\varepsilon, \varepsilon; 1-3\varepsilon, -\varepsilon; -z) \right). \quad (\text{D.70})$$

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